The turbulent birth of planets

Wlad Lyra

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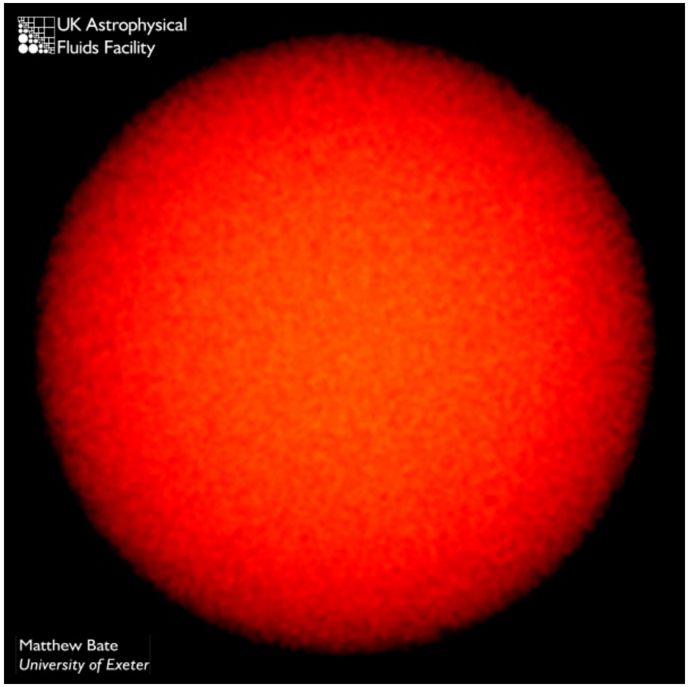
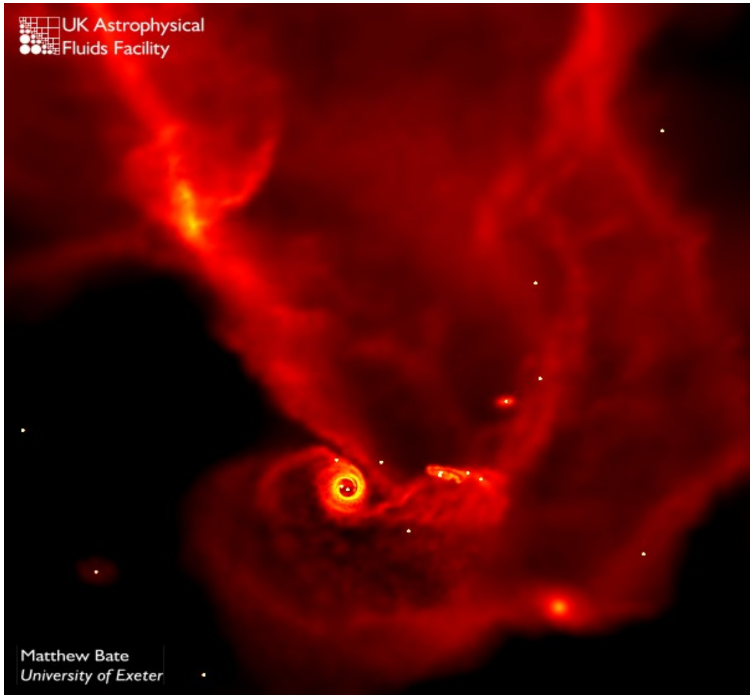
Star Formation - The B3 simulation (Bate, Bonnell, Bromm 2003)



t=0



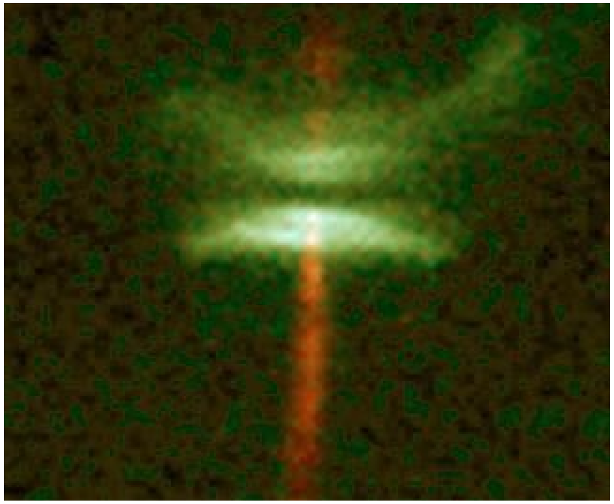
t=266 000 yr

Some stars are seen to be born with lots of surrounding gas.

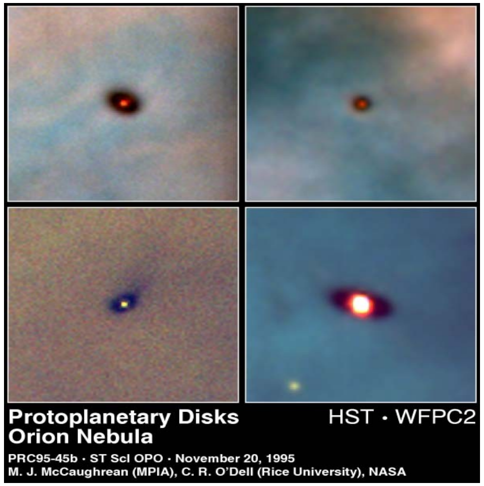
This gas is bound to the star and referred to as

circumstellar disk or *protoplanetary disk.*

“Extra-Solar Nebulae” - Circumstellar Disks

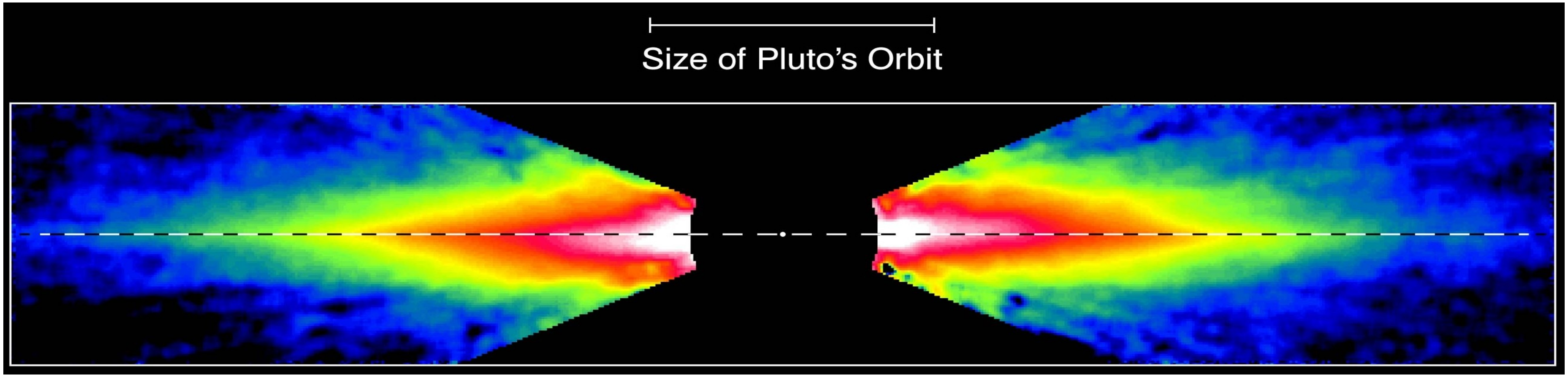


Dust lane
blocks view



A light background
reveals the disks

Face-on disk
in reflected light



The disk of Beta Pictoris

Accretion

"The central problem
of nearly 30 years of accretion disk theory
is to understand how they accrete"

Source: Balbus & Hawley 1998

Accretion time for molecular viscosity

$$\nu : cm^2 s^{-1} \qquad \nu = l^2 / t_{coll} = l V_t \qquad t_{acc} = r^2 / \nu$$

For a newly formed disk

$$r = 10^{14} cm ; n = 10^{15} cm^{-3} ; \sigma = 10^{-16} cm^{-2}$$
$$l = 1/(n\sigma) = 10 cm ; V_t = 10^5 cm s^{-1}$$

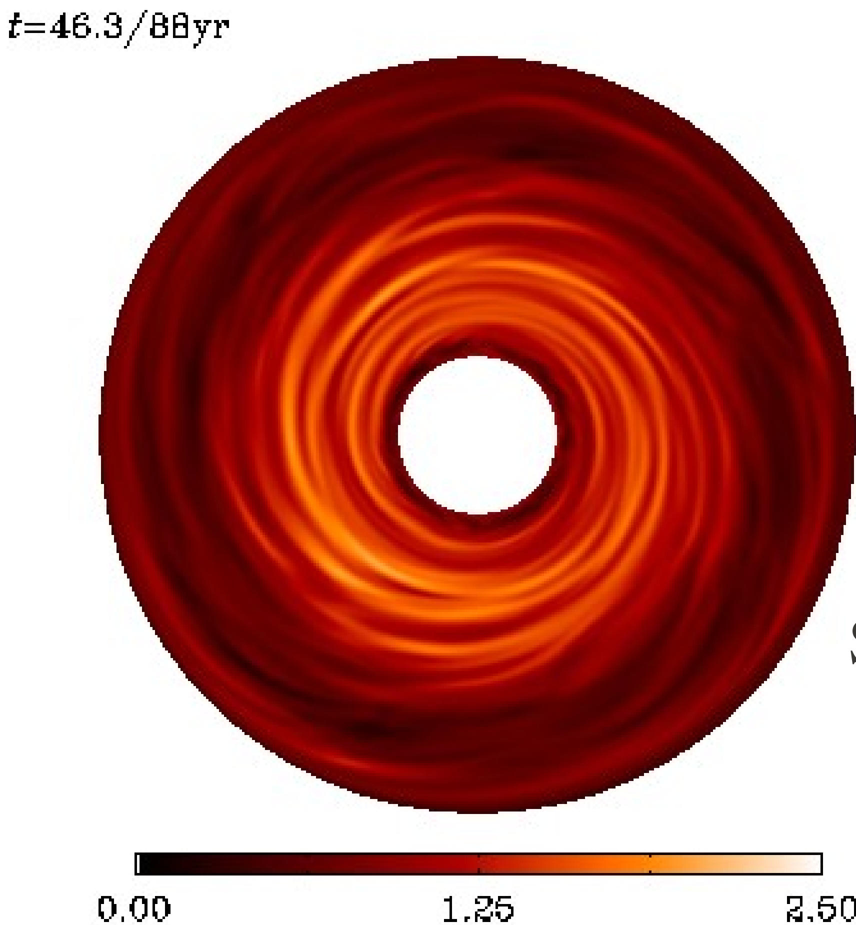
$$t_{acc} = 10^{13} yr$$

Observations reveal that disks only live up to 10^7 – 10^8 yr

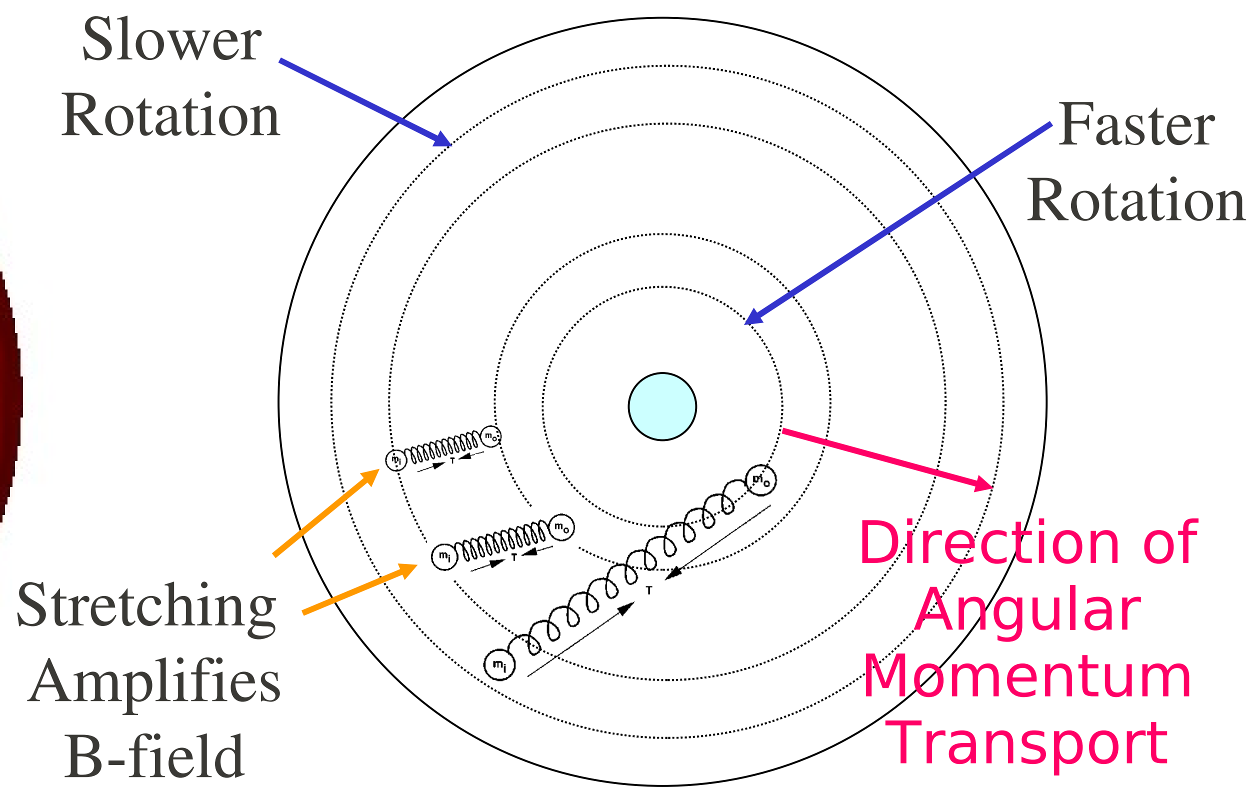
Much more powerful viscosity needed!

Accretion needs turbulent viscosity

Turbulence in disks is enabled by
the Magneto-Rotational Instability

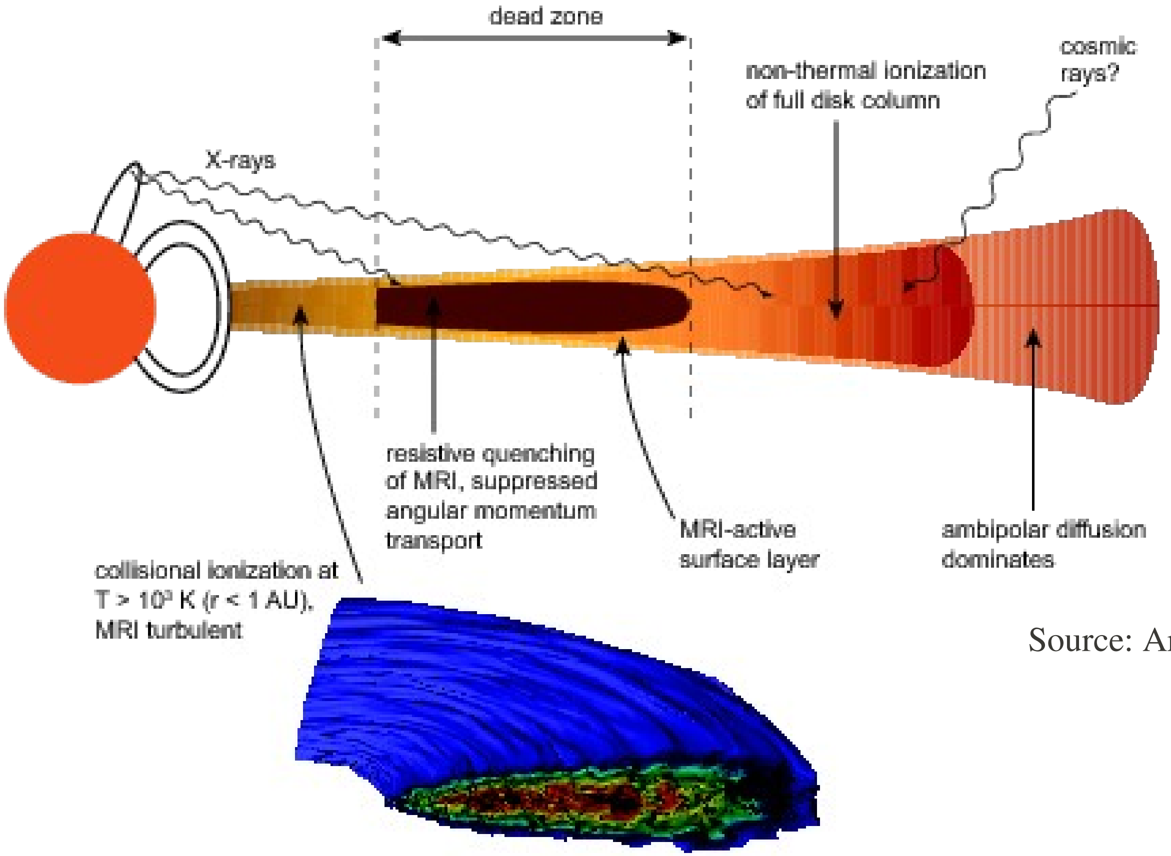


MRI sketch



Unstable if angular velocity decreases outward

Dead zones are robust features of accretion disks

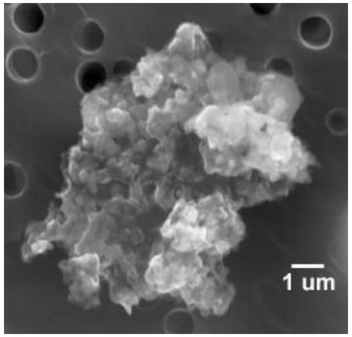


Source: Armitage (2010)

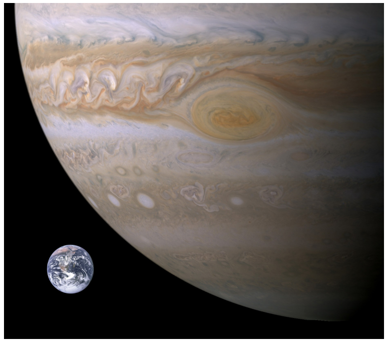
A substantial portion of the disk
is of **low ionization**
and therefore **dead** to the MRI

Planet Formation

"Planets form in disks of gas and dust"



A miracle happens



Planet Formation

Planetesimal Hypothesis (Safronov 1969)

From dust to boulders

μm -> m: van der Wall forces cause sticking

m -> km: A miracle happens

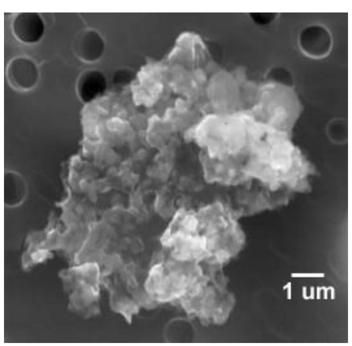
From planetesimals to protoplanets

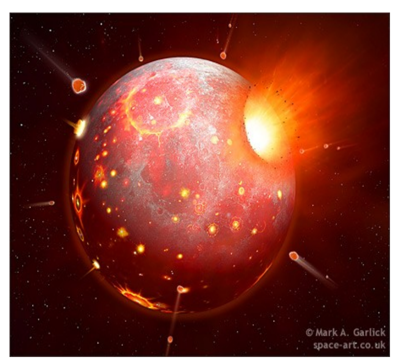
km -> 1000 km: Gravity

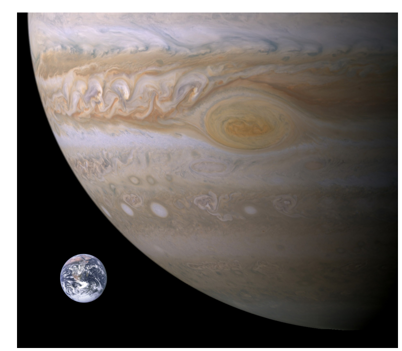
From protoplanets to planets

Rocky Planets: Protoplanets collide

Gas Giants: Attract gaseous envelope







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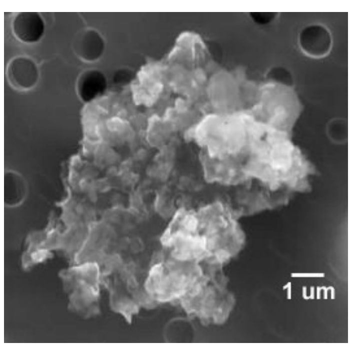
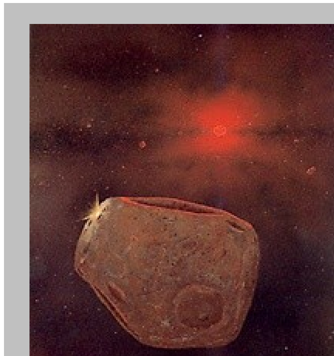
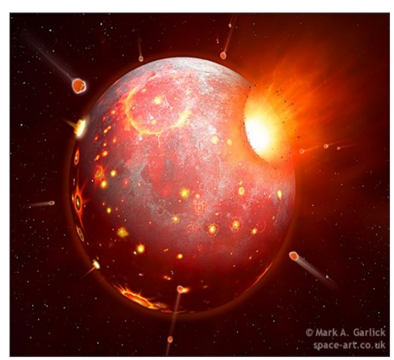






The meter size barrier

Growth barrier

- through EM?

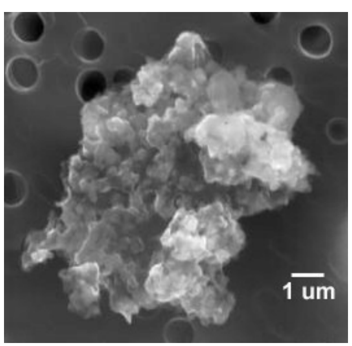
They don't stick, they break

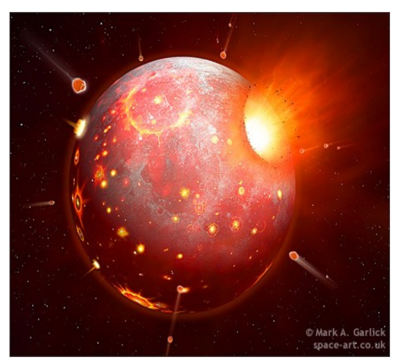
- through Gravity?

They aren't massive enough

Timescale barrier

They migrate quite fast





The meter size barrier

Growth barrier

- through EM?

They don't stick, they break

Gentle Collisions

- through Gravity?

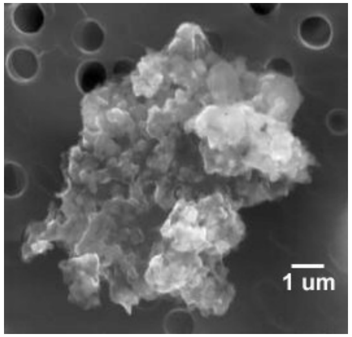
They aren't massive enough

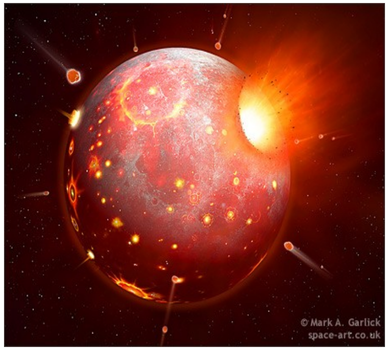
High number density

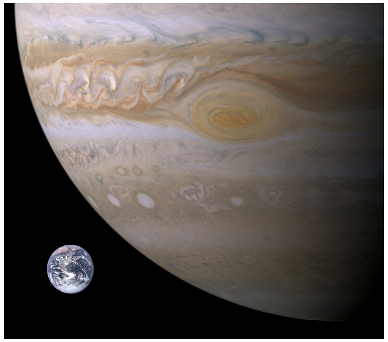
Timescale barrier

They migrate quite fast

Stopping Mechanism



All you need to know about aerodynamics

Gas $$\frac{D\boldsymbol{u}}{Dt} = -\nabla\Phi - \rho^{-1}\nabla p$$

Solids $$\frac{d\boldsymbol{w}}{dt} = -\nabla\Phi - \frac{(\boldsymbol{w}-\boldsymbol{u})}{\tau}$$

$\boldsymbol{V} = \boldsymbol{w} - \boldsymbol{u}$

$$\frac{D\boldsymbol{V}}{Dt} \approx \rho^{-1}\nabla p - \frac{\boldsymbol{V}}{\tau}$$

$$V_t = \tau\rho^{-1}\nabla p$$

The drag force pushes
the solids ***towards*** the pressure gradient

solids move toward pressure maxima

Solids in a turbulent disk

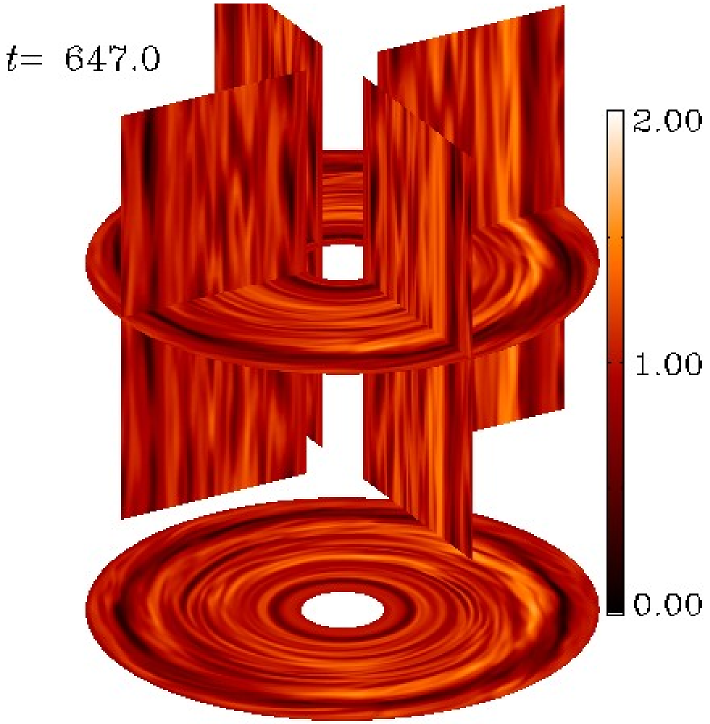
Gas $$\frac{D\boldsymbol{u}}{Dt} = -\nabla\Phi - \rho^{-1}\nabla p$$

Solids $$\frac{d\boldsymbol{w}}{dt} = -\nabla\Phi - \frac{(\boldsymbol{w}-\boldsymbol{u})}{\tau}$$

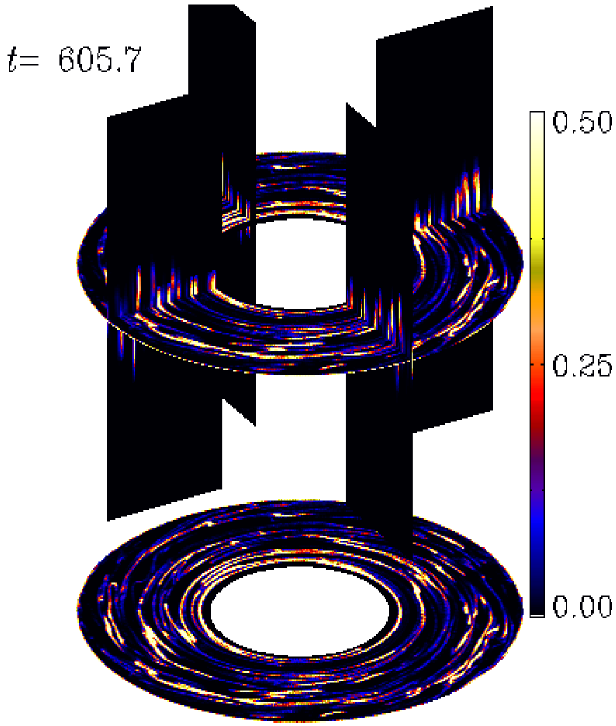
$$\boldsymbol{w} = \boldsymbol{u} + \tau\rho^{-1}\nabla p$$

The drag force
pushes the solids *towards*
the pressure gradient

Gas



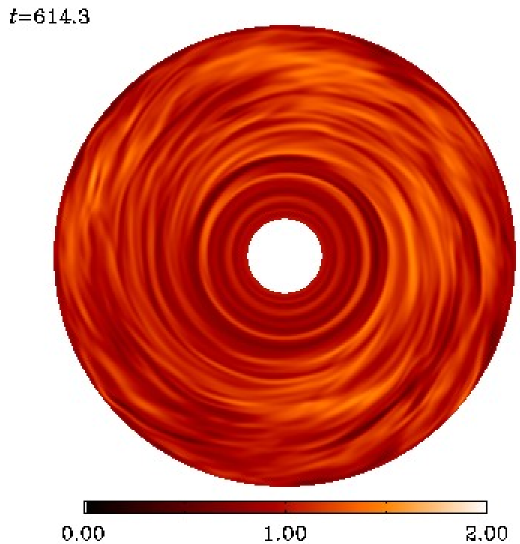
Solids

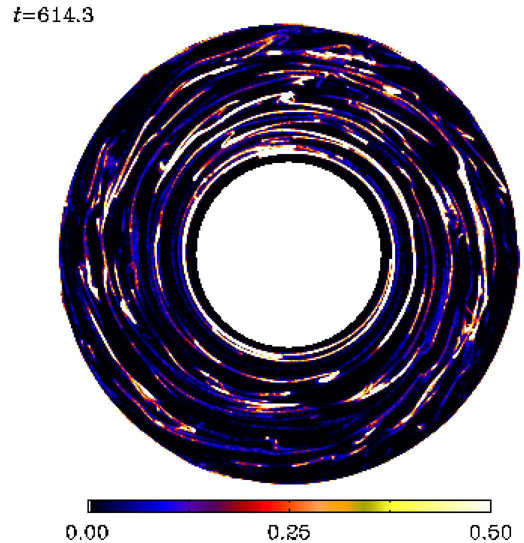


Intense Clumping!!

Source: Lyra et al. (2008a)

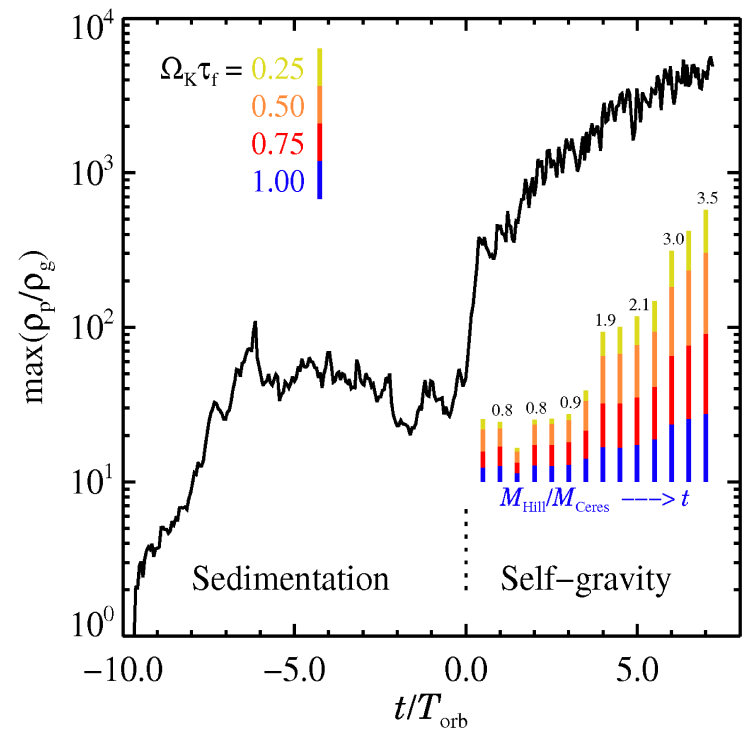
Solids in a turbulent disk





Source: Lyra et al. (2008a)

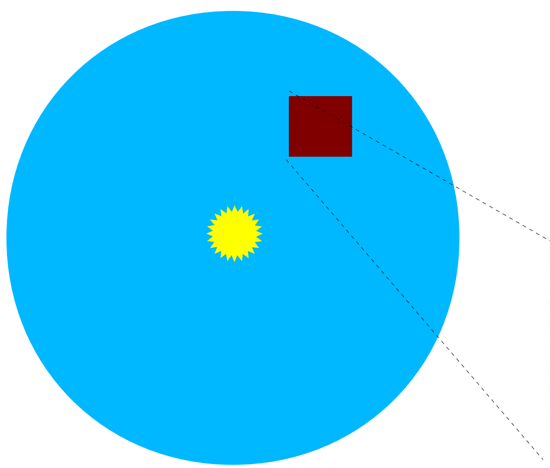
Breaching the meter size barrier by a giant leap



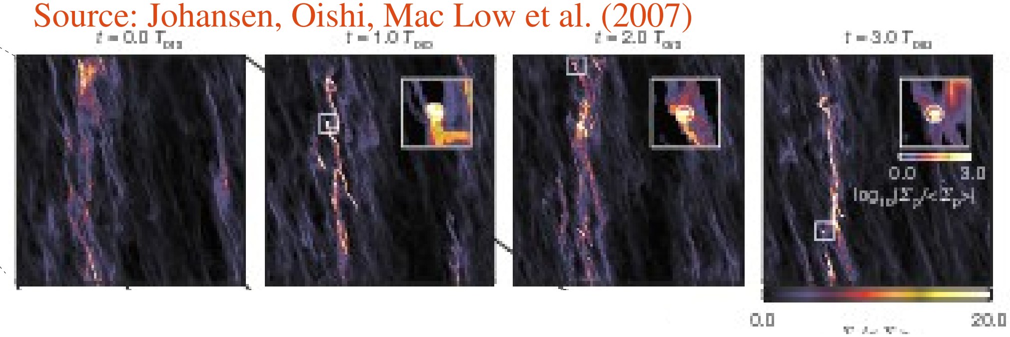
-Turbulent eddies are very efficient particle traps

-Correlation between gas and solids density maxima

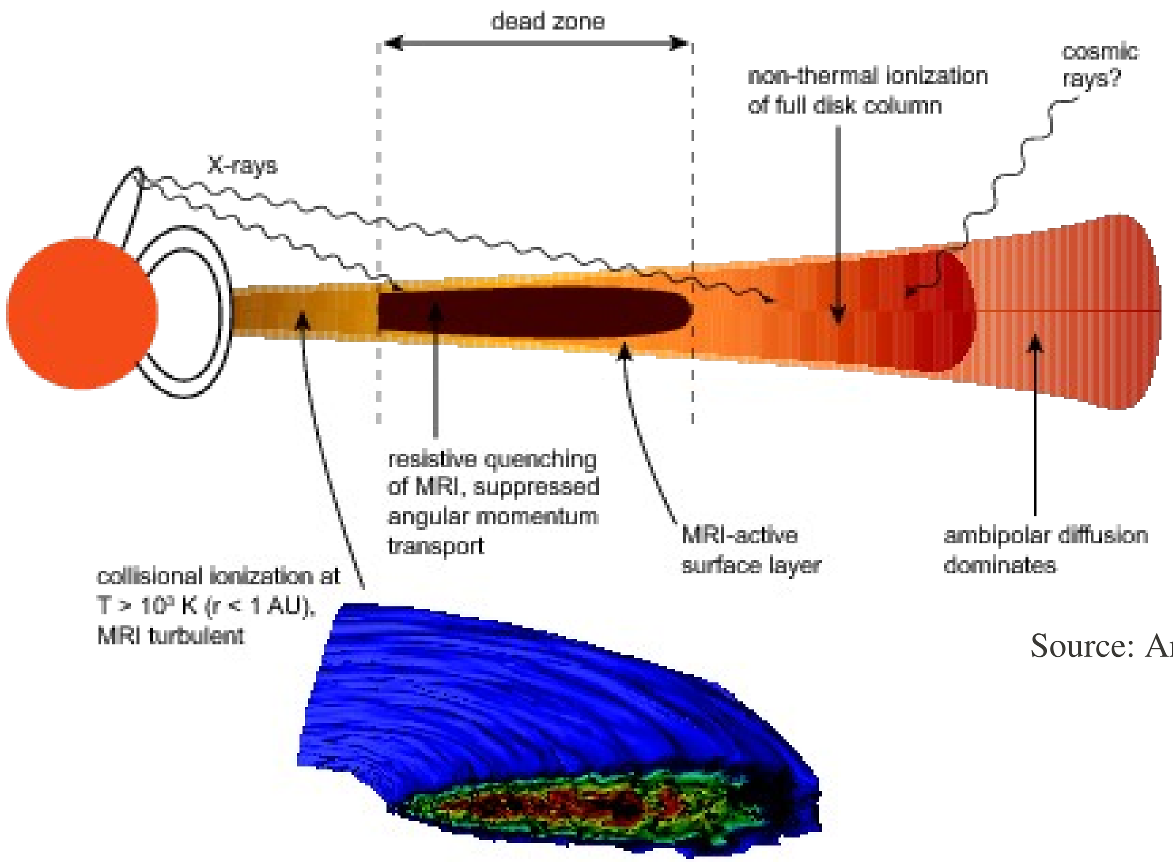
- Critical density for gravitational collapse of clumps



Source: Johansen, Oishi, Mac Low et al. (2007)



Alas... Dead zones are robust features of accretion disks

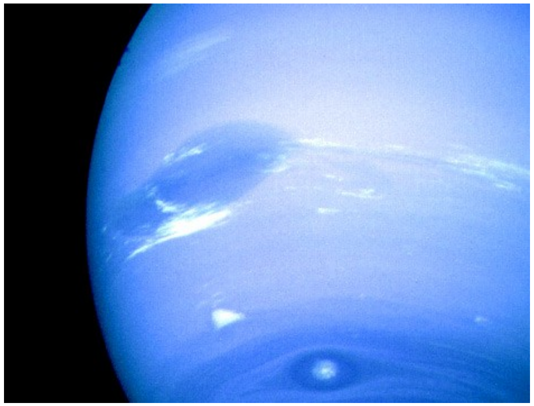


Source: Armitage (2010)

The planet formation region
(inner 10AU)
falls squarely within the dead zone

Therefore....
The search for hydrodynamical routes
for turbulence continues.

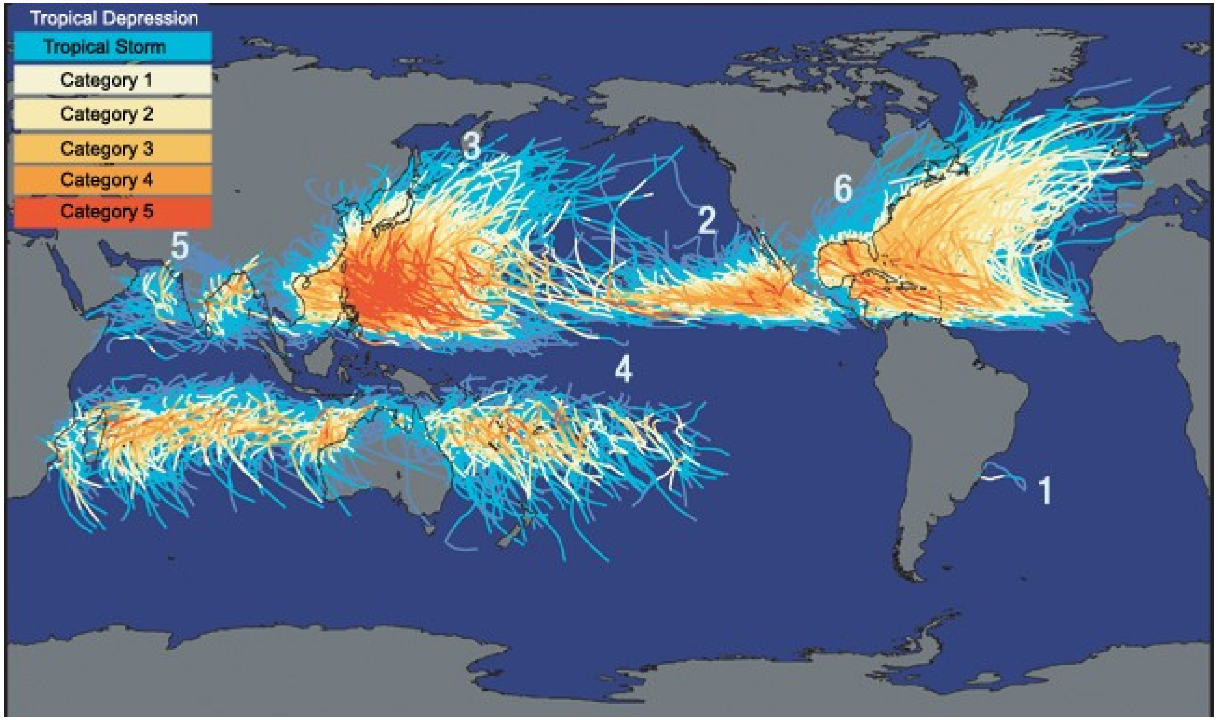
A possibility: Baroclinic Instability

- Well known in planetary atmospheres
- Leads to the formation of **vortices**

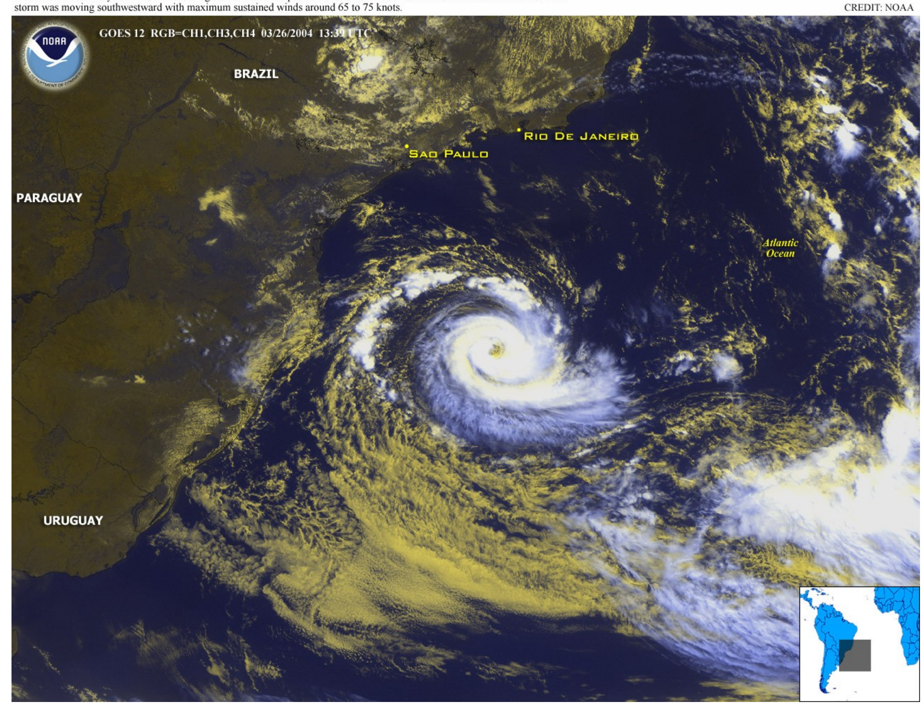
And vortices are:

- A solution of the NS equations: persistent structures
- Very interesting for planet formation

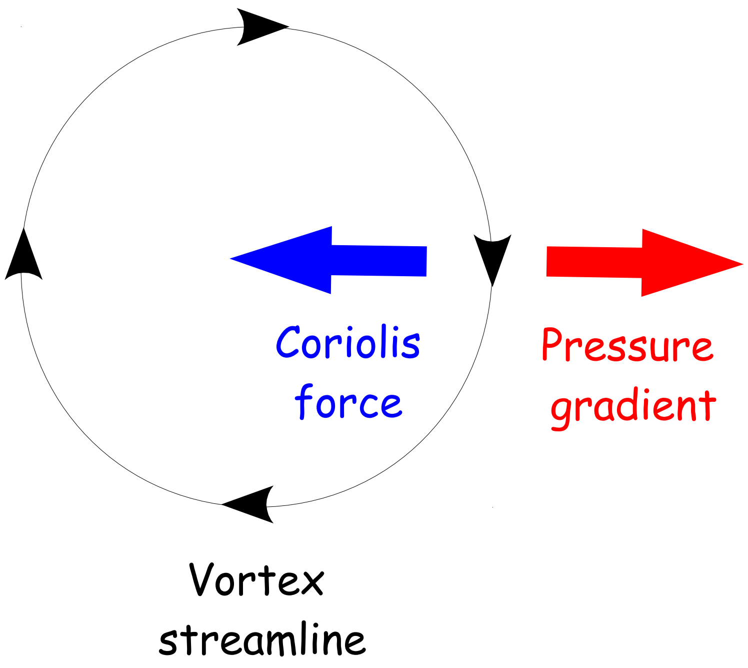


This GOES-12 image shows an extremely unusual tropical storm that was located over the South Atlantic Ocean on 03/26/2004. NOAA Satellite Analysis Branch meteorologists estimated the position of the storm was near 28.9S 43.7W at 11:45 UTC. The storm was moving southwestward with maximum sustained winds around 65 to 75 knots.

CREDIT: NOAA



Vortex Equilibrium

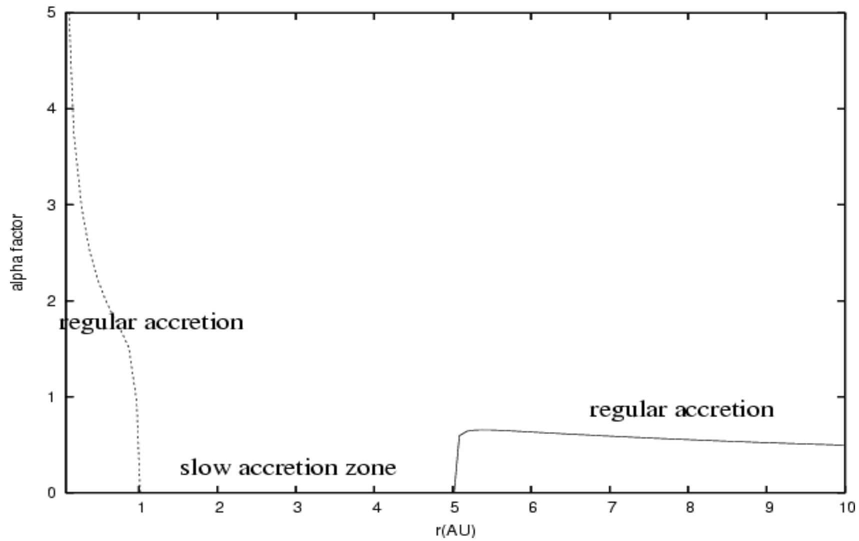


Geostrophic balance:

$$2\Omega\times\boldsymbol{v}=-\rho^{-1}\nabla p$$

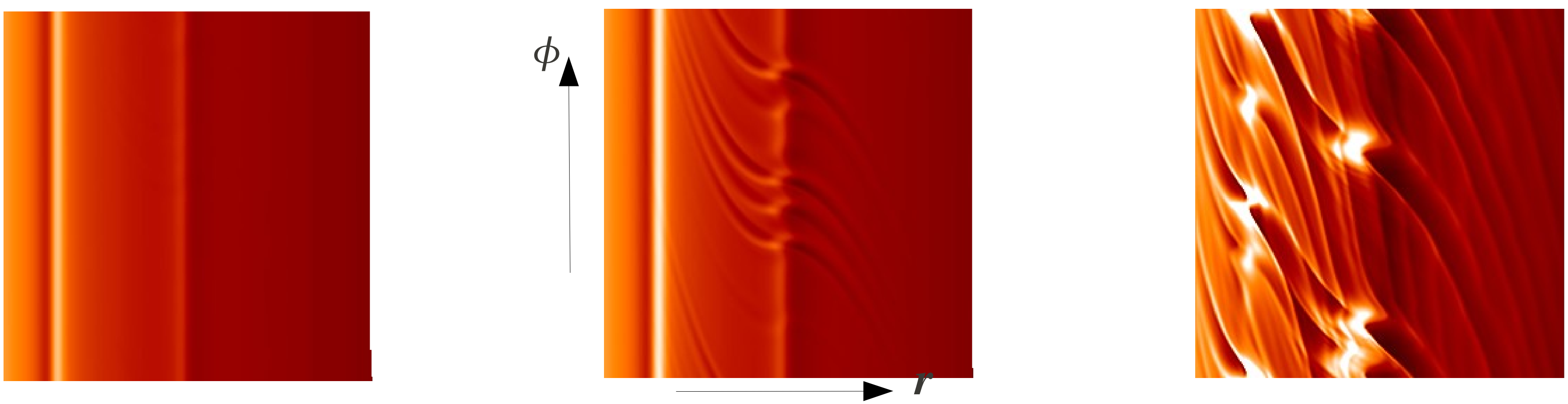
- Particles do not feel the pressure gradient.
- They just sink towards the center, where they accumulate.
- Aid to planet formation (von Weizsäcker, 1946)
- Revisited by Barge & Sommeria (1995)

A simple Dead Zone model



Alpha-disk with viscosity jumps

Source: Varniere & Tagger (2006)



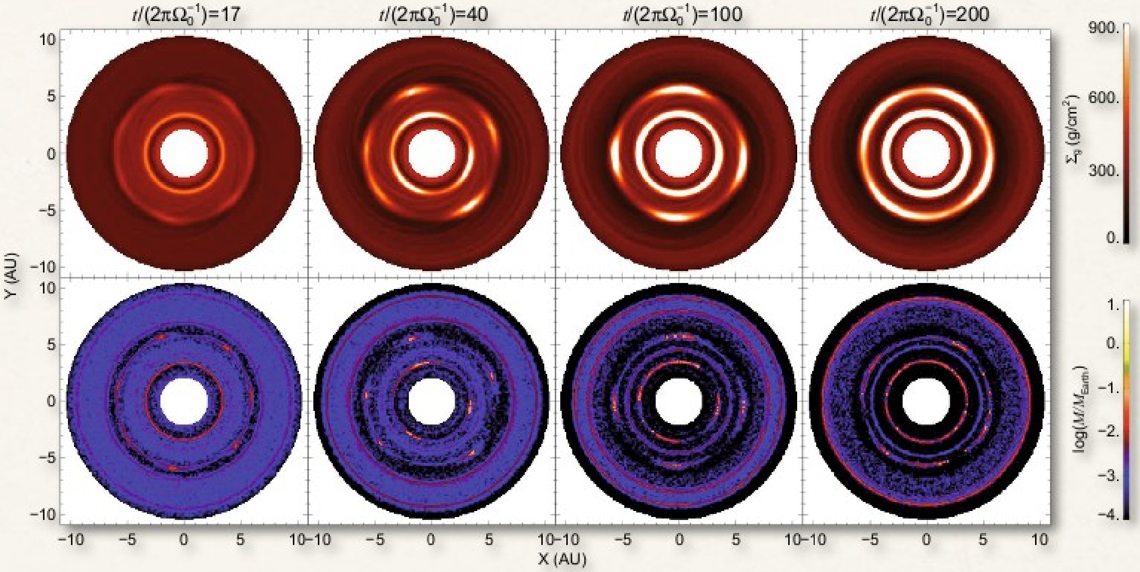
Inflow discontinuity triggers the Rossby wave instability (RWI)...

...that saturates into vortices

Source: Lyra et al. (2008b)

Planet formation and vortices

- Vortex generated near a dead zone: «planets» produced in ~5 orbits (Lyra *et al.* 2008a).

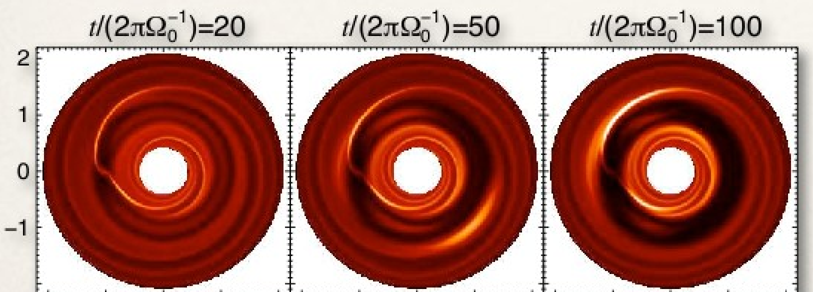


- Planetary gap vortices (Lyra *et al.* 2008b).

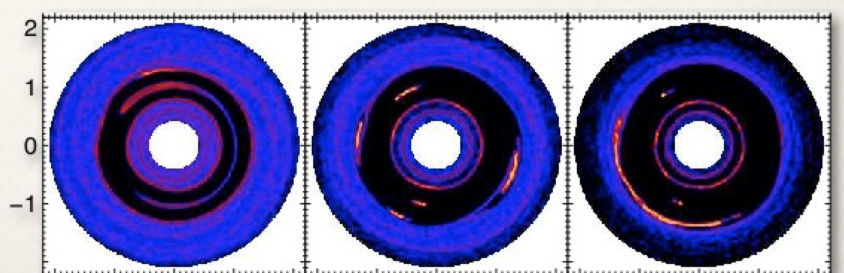
Gas

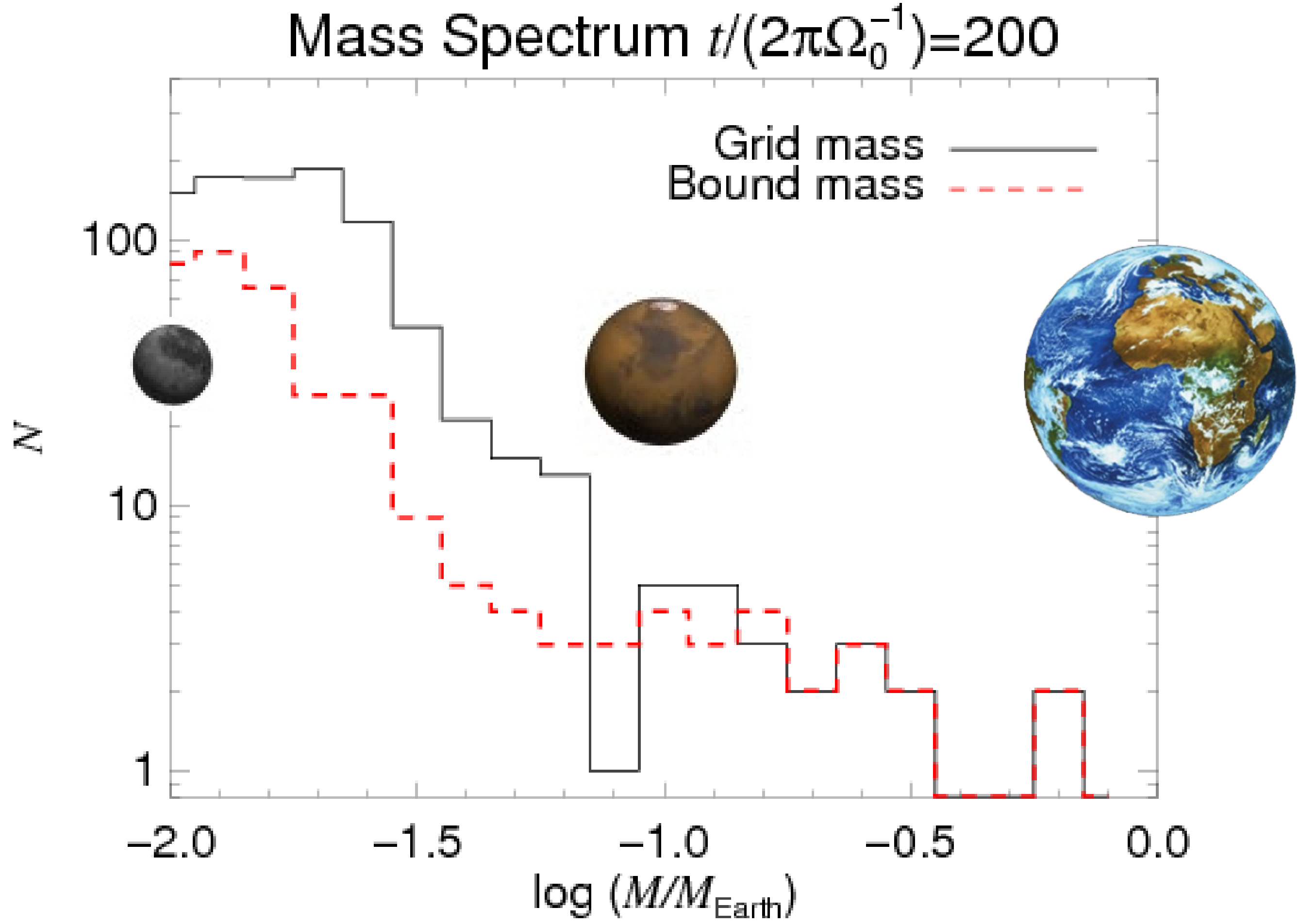
Particles

Gas



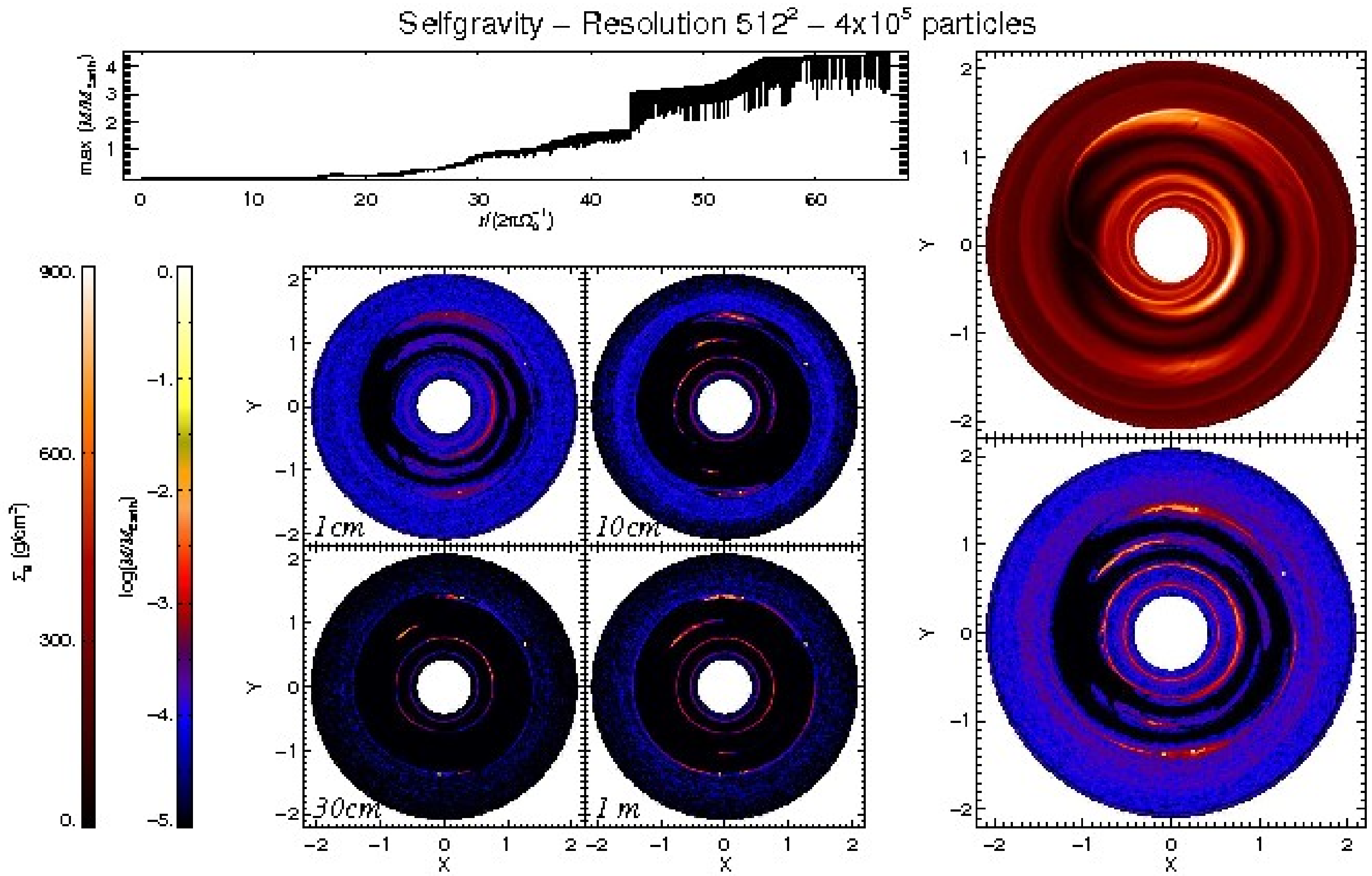
10 cm



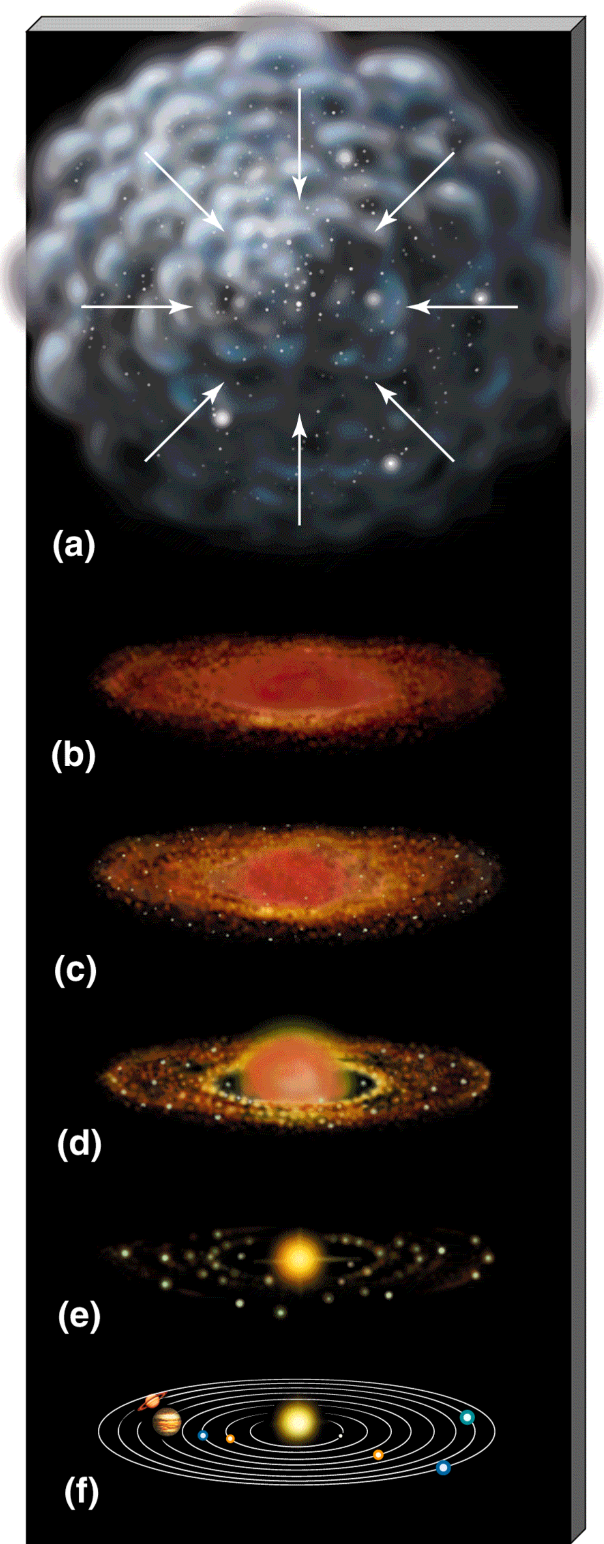


- Mass spectrum by the end of the simulation
- 300 bound clumps were formed
- Power law d(log N)/d(log M)=-2.3 +/- 0.2
- 20 of these are more massive than Mars

Vortex trapping



3 super-Earths formed + Mars mass Trojans



Summarizing

Gravitational collapse of an interstellar cloud

Outward transport of angular momentum through turbulence generated by the MRI. Dust coagulates into pebbles and boulders, sedimenting towards the midplane.

Rocks in the turbulent medium are trapped in transient pressure maxima and undergo collapse into planetesimals and dwarf planets.

Vortices may be excited in the dead zone. Inside them, the first dozens of Mars-mass embryos are formed.

Embryos collide and give rise to the oligarchs (?)

When Jupiter is formed, a second round of planet formation is triggered (Trojan planets, Saturn?)

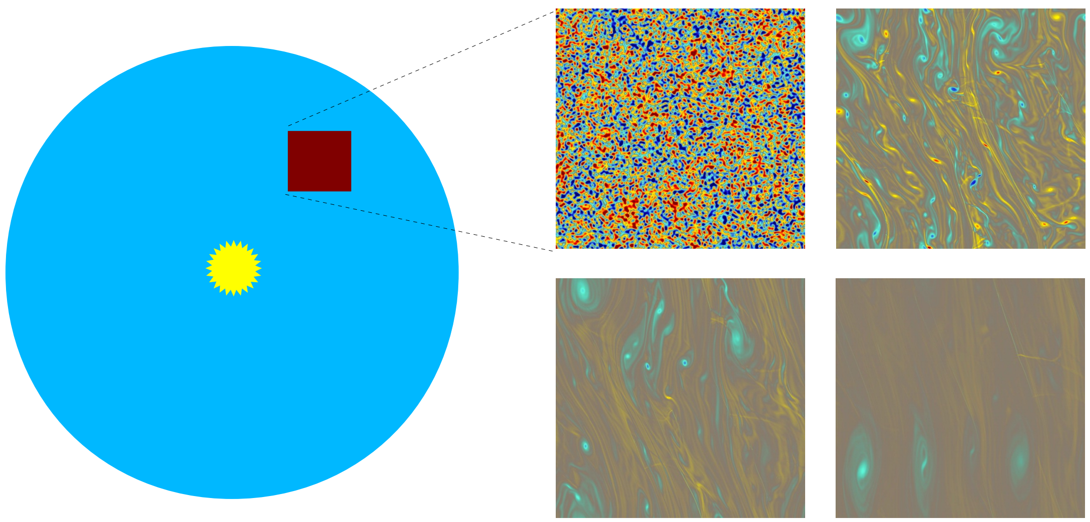
Nice model: Jupiter and Saturn cross 2:1 MMR and define the architecture of the Solar System

Vortex Stability

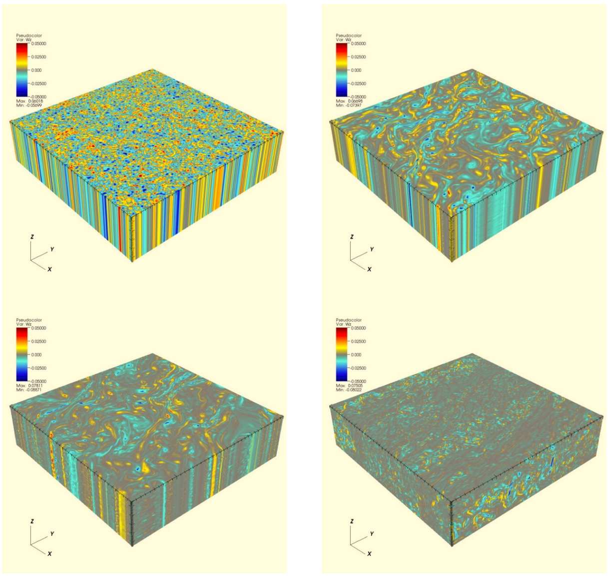
Words of a referee...

"The formation mechanism presented depends on the presence of ***vortices***, the existence of which is well known to be ***problematic in 3D*** (Barranco & Marcus 2005; Shen, Stone & Gardiner 2006)."

Shen et al. (2006) – Unstructured Noise



2D
Inverse Cascade



3D
No Inverse Cascade

Vortex Stability – Much Ado About Nothing

NS: $$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\Phi - \rho^{-1}\nabla p + \nu\nabla^2\boldsymbol{u} \qquad (\nabla\times \quad)$$

Vorticity Equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - \boldsymbol{\omega}(\nabla\cdot\boldsymbol{u}) + (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\boldsymbol{\omega}$$

Vortex Stability – Much Ado About Nothing

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Vorticity Equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}}_{\textbf{advection}} \underbrace{-\boldsymbol{\omega}(\nabla\cdot\boldsymbol{u})}_{\textbf{compression}} + \underbrace{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}_{\textbf{stretching}} + \frac{1}{\rho^2}\underbrace{\nabla\rho\times\nabla p}_{\textbf{baroclinicity}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\textbf{dissipation}}$$

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The baroclinic term is the **only** source term!

Barotropic x Baroclinic

Vorticity Equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}}_{\text{advection}} \underbrace{-\boldsymbol{\omega}(\nabla\cdot\boldsymbol{u})}_{\text{compression}} + \underbrace{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}_{\text{stretching}} + \frac{1}{\rho^2}\underbrace{\nabla\rho\times\nabla p}_{\text{baroclinicity}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\text{dissipation}}$$

Barotropic equation of state: ***p=p(ρ)***

Trope = direction (e.g. isotropic)
Baro + tropic = pressure gradient same direction as the density gradient

Baroclinic equation of state: ***p=p(ρ, T)***

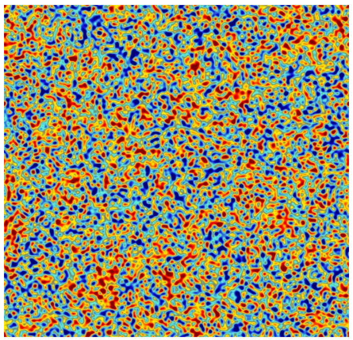
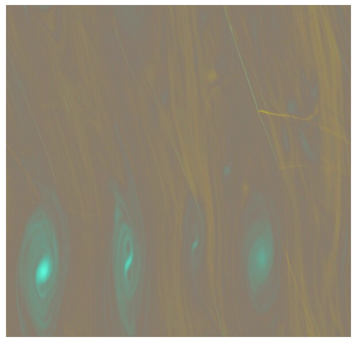
Cline = inclination (e.g. tachocline)
Baro + clinic = pressure and density planes inclined

Vortex Stability – Much Ado About Nothing

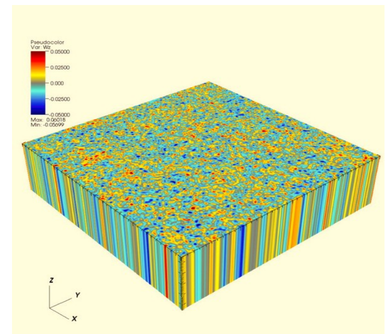
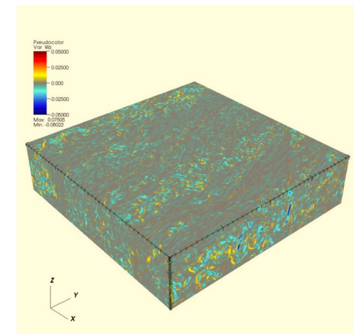
Shen et al. (2006) – Local Box

$p=p(\rho)$ - Pressure is a function of density only

NO BAROCLINICITY

2D – No stretching

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Vortex Stability – Much Ado About Nothing

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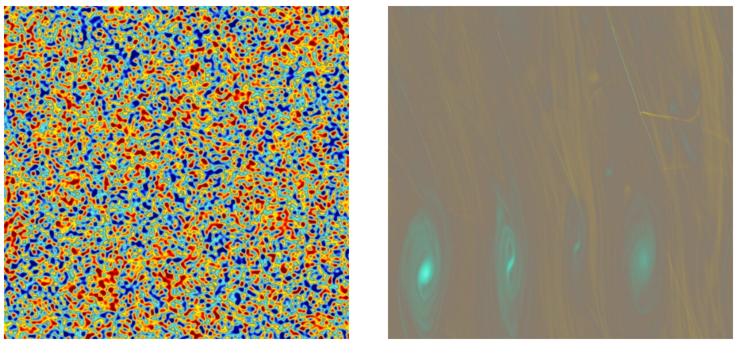
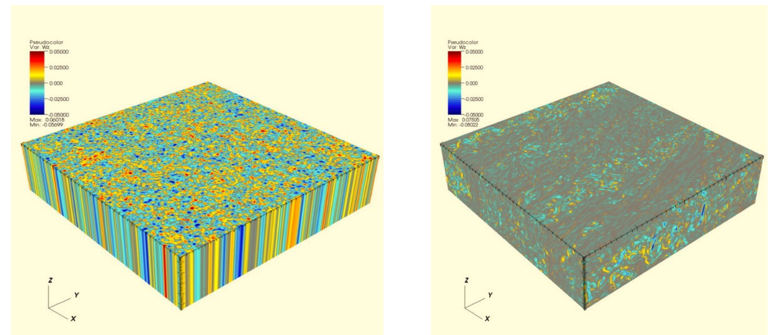


2D – No stretching



3D - Stretching

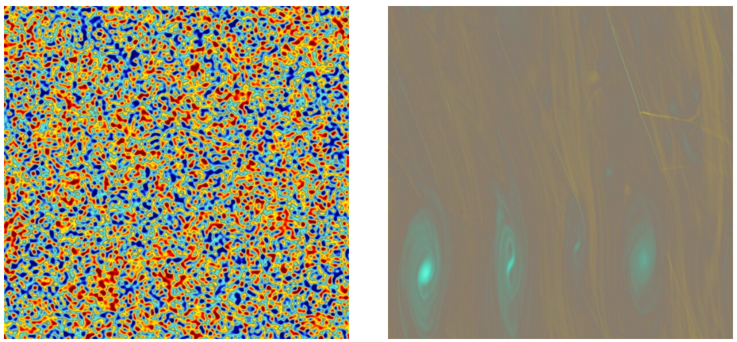
$$\frac{D\boldsymbol{\omega}}{Dt} = \underbrace{-\boldsymbol{\omega}(\nabla\cdot\boldsymbol{u})}_{\textbf{compression}} + \underbrace{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}_{\textbf{stretching}} + \cancel{\underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\textbf{baroclinicity}}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\textbf{dissipation}}$$

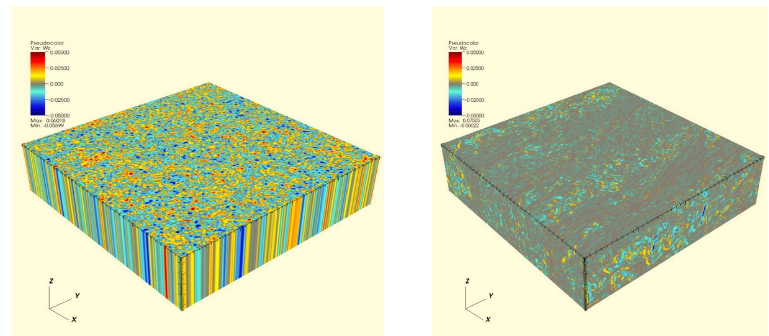
Vortex Stability – Much Ado About Nothing

Shen et al. (2006) – Local Box

$p=p(\rho)$ - Pressure is a function of density only

NO BAROCLINICITY





2D – No stretching

3D - Stretching

$$\frac{D\boldsymbol{\omega}}{Dt} = 0$$

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}$$

Without baroclinity, nothing counters the stretching term

No surprise that the vortices decay...

Baroclinic Instability – Excitation and self-sustenance of vortices

The situation changes considerably when including an ***entropy gradient***.

Why entropy?

Entropy: $s = p / \rho^{\gamma}$

Polytropic eq. of state: $p = k\, \rho^{(n+1)/n}$

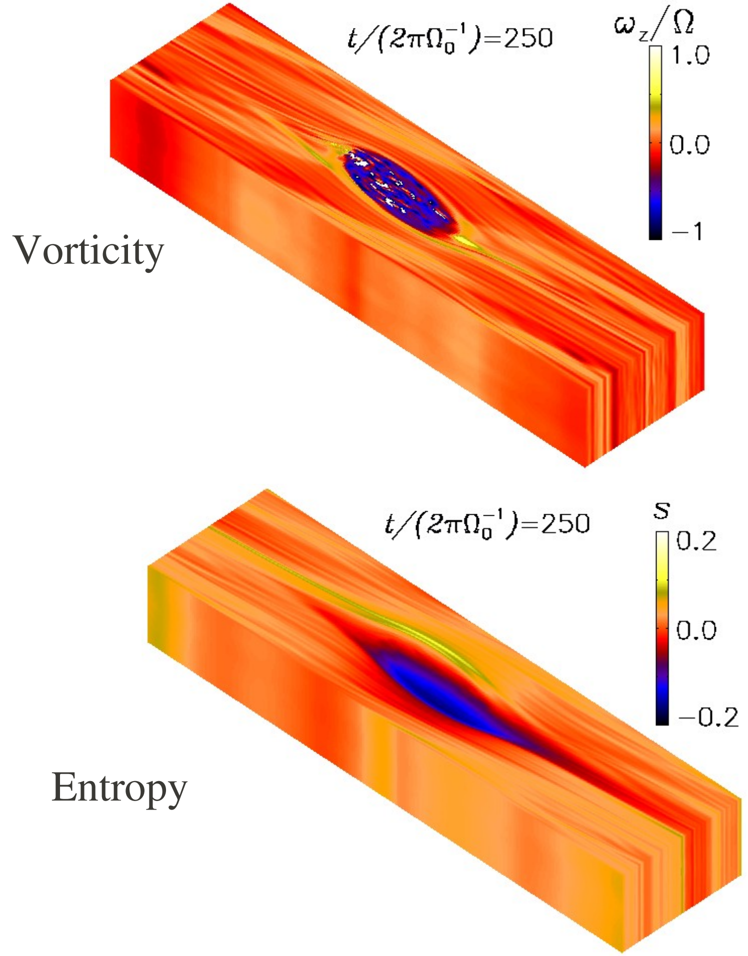
For $n = \dfrac{1}{\gamma - 1}$; $s = k$

Entropy is the constant in the polytropic equation of equation

Therefore:

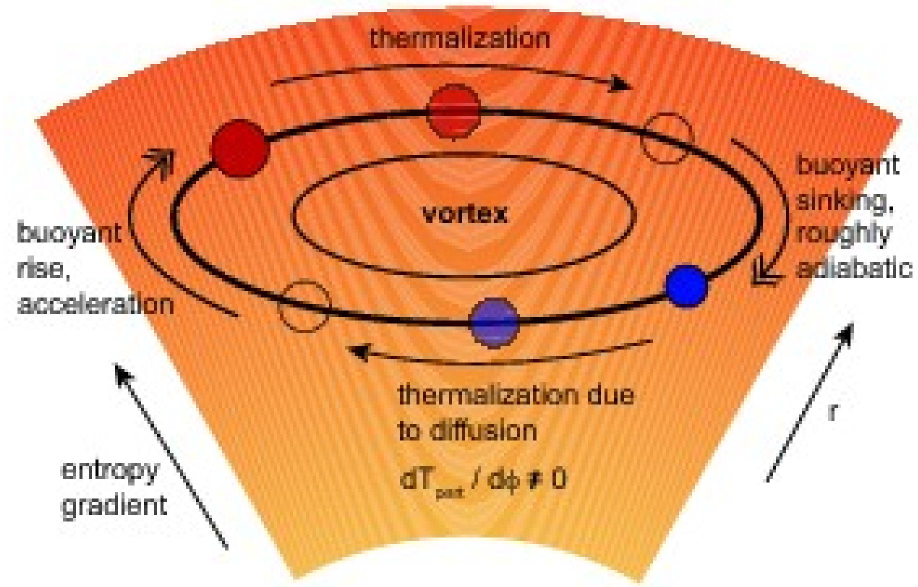
A spatial gradient of entropy translates into a departure from barotropic conditions

Baroclinic Instability – Excitation and self-sustenance of vortices



Source: Lyra & Klahr (2010)

Sketch of the Baroclinic Instability

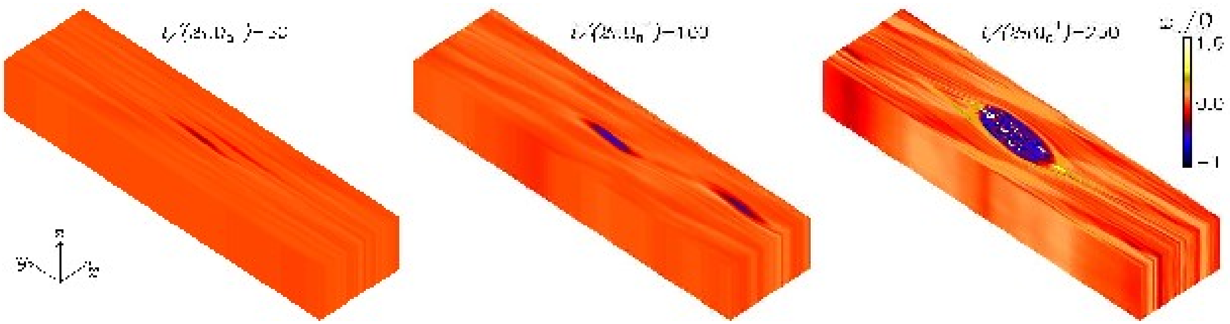


Source: Armitage (2010)

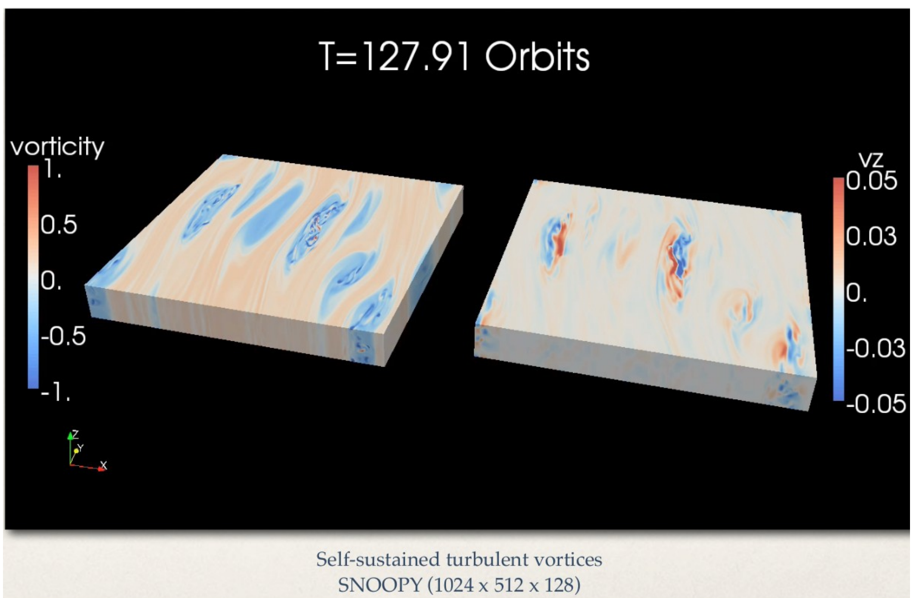
Baroclinic Instability – Excitation and self-sustenance of vortices

The Baroclinic Instability in three dimensions

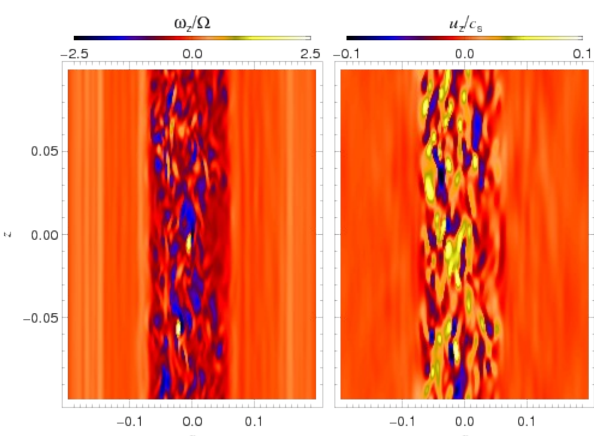
Vorticity



Source: Lyra & Klahr (2010)

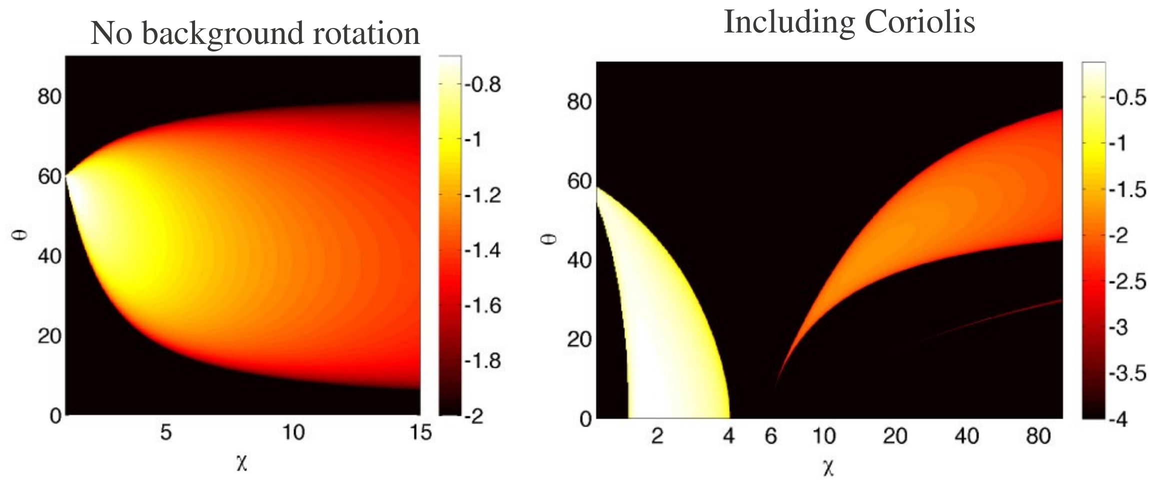


Source: Lesur & Papaloizou (2010)

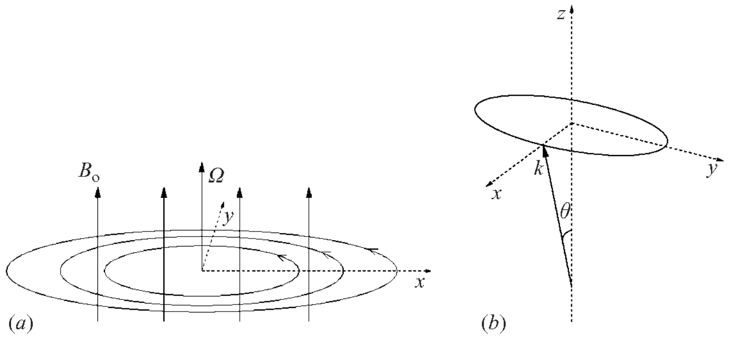


Vorticity Kin. Energy

Elliptic Instability



Source: Lesur & Papaloizou (2009)



Source: Mizerski & Bajer (2009)

Instability of elliptic streamlines

* In the **non-rotating** case:

- **Resonance** between

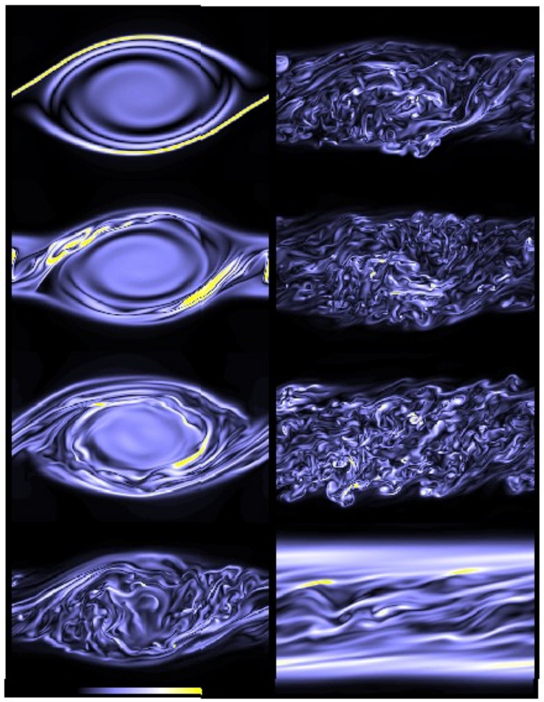
  vortex turnover frequency and **inertial waves**

* In the **rotating** case:

- Strong "horizontal" (theta=0) unstable mode:

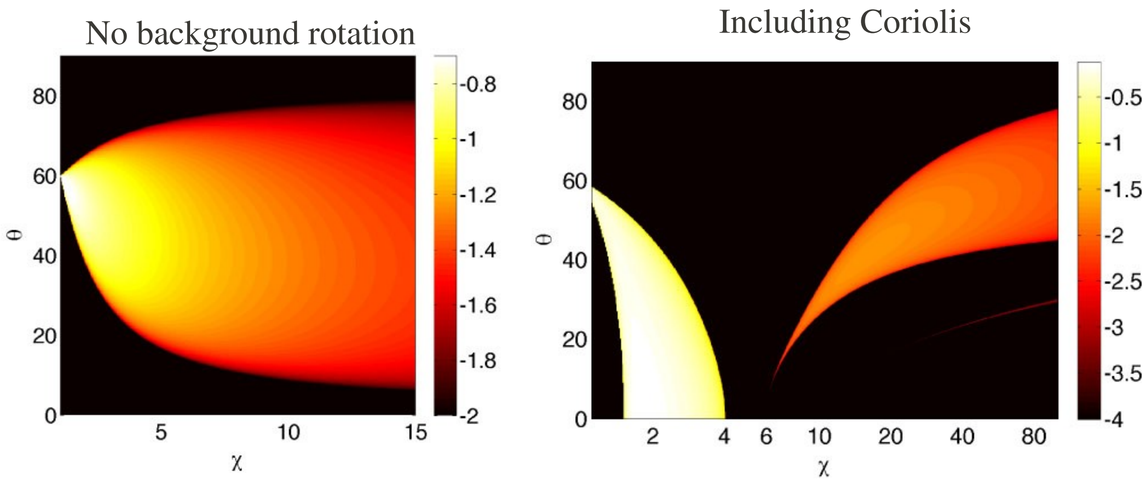
  Exponential growth of epicyclic disturbances

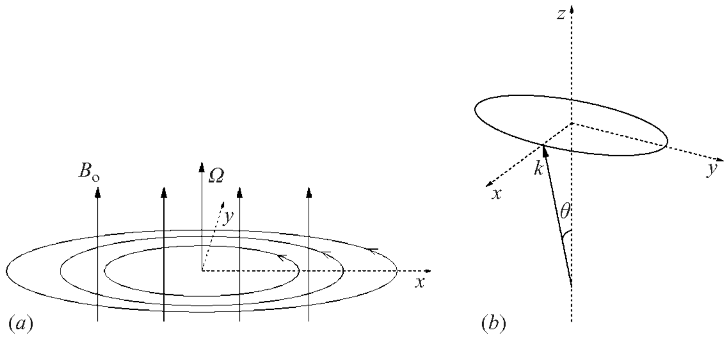
Vortex coherence is destroyed
Energy cascades forward and dissipates



Source: McWilliams (2010)

Elliptic Instability





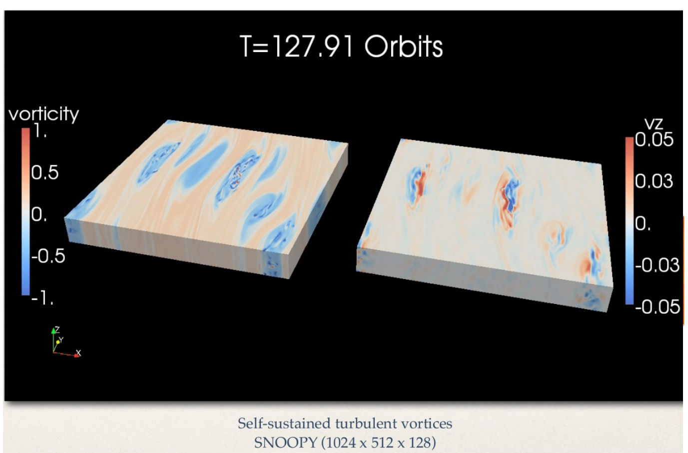
Source: Lesur & Papaloizou (2009)

Source: Mizerski & Bajer (2009)

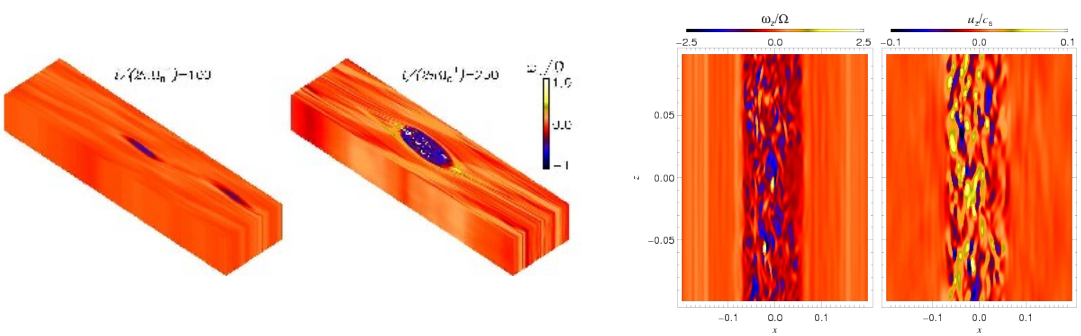
Despite the elliptical instability,

baroclinity keeps the vortex coherent.

The result is "core turbulence" only



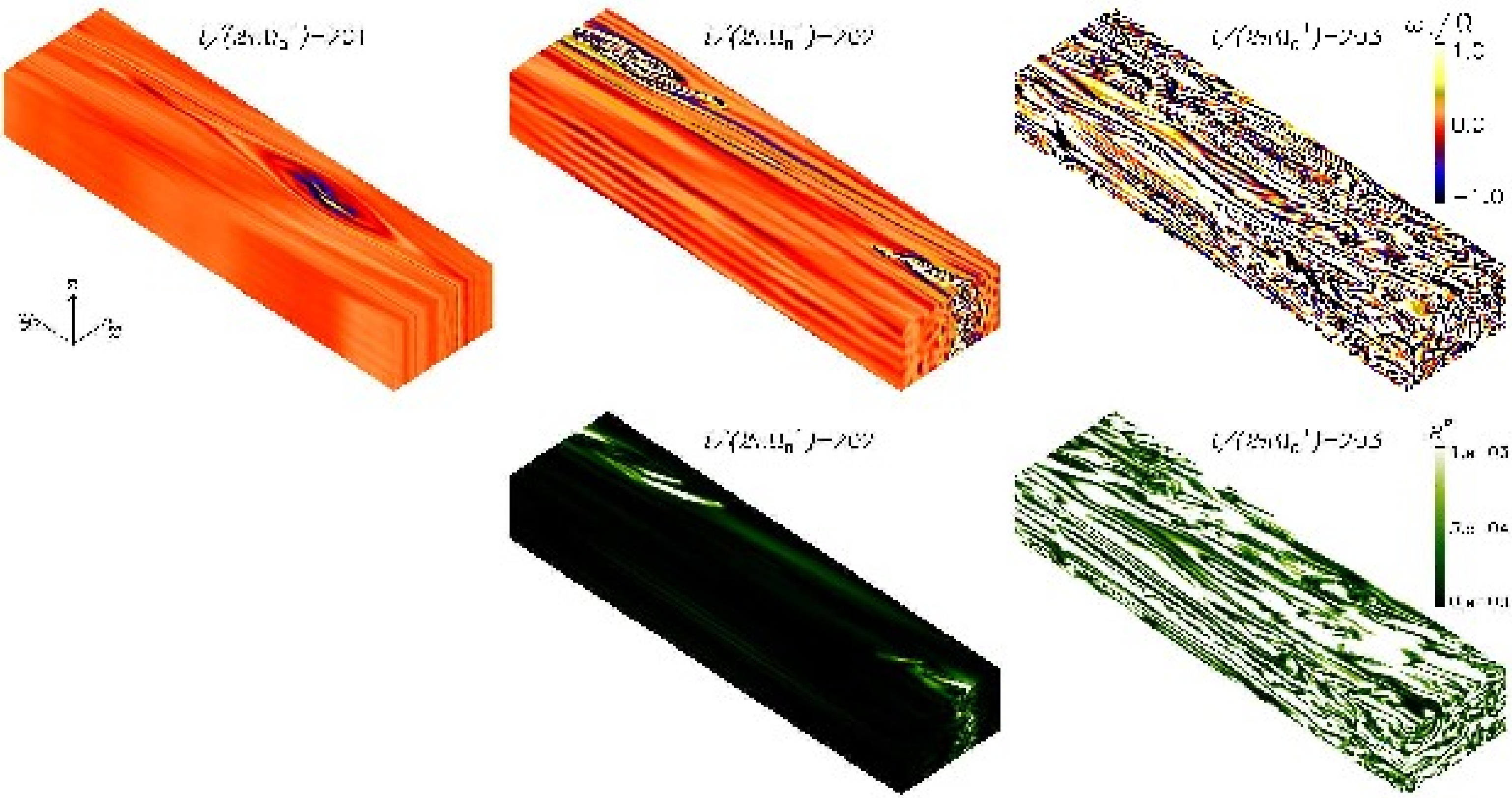
Source: Lesur & Papaloizou (2010)



Source: Lyra & Klahr (2010)

Interaction of Baroclinic and Magneto-Rotational Instabilities

What happens when the vortex is magnetized?



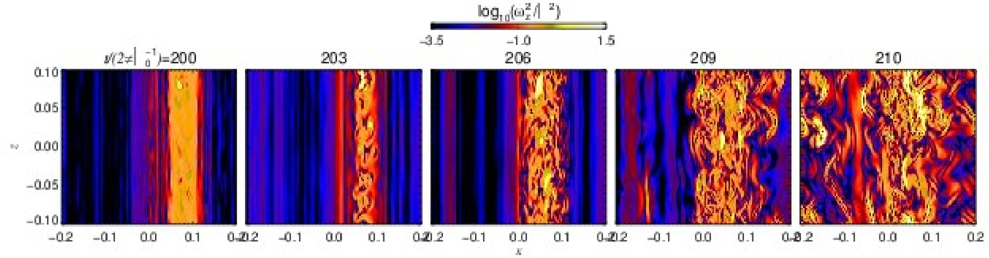
Vorticity

Magnetic Energy

Vortex gone!

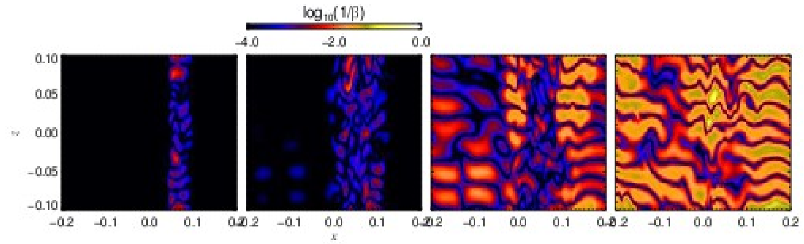
Vortex MHD instability?

Notice that the vortex goes turbulent *before* the box.

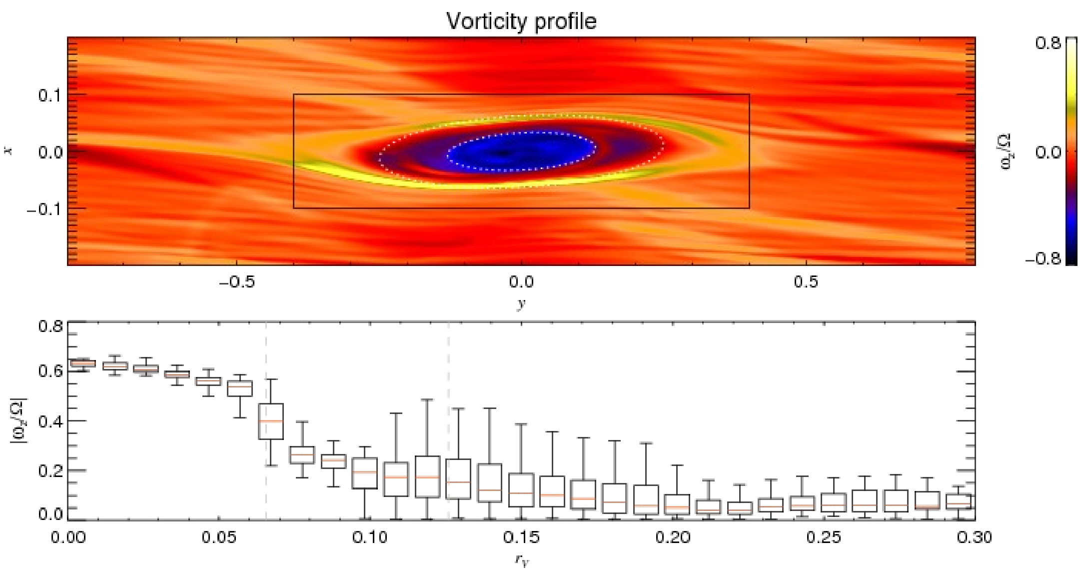


Enstrophy

Is this the MRI?



Mag. Energy



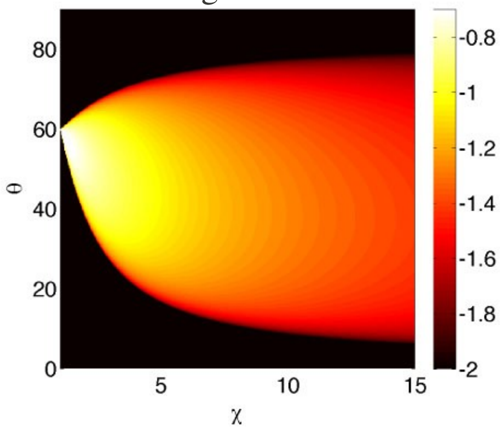
The MRI needs shear.
Yet, the core rotates close to rigid.

*So, **NO**, this is **not** the MRI.*

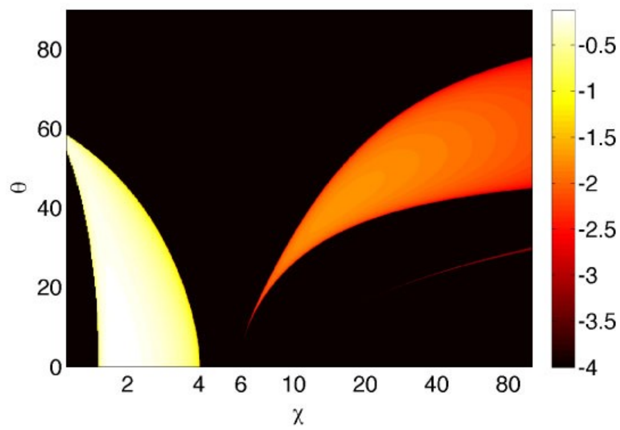
A magnetoelliptic mode?

Magneto-Elliptic Instability

No background rotation



Including Coriolis



Including Lorentz

?

Mizerski & Bajer (2009, Journal of Fluid Mechanics)

"The presence of magnetic fields widens the range of existence of the horizontal instability to an unbounded interval of aspect ratios when

$$Ro^{-1} < -\frac{b^2}{4}$$

Magneto-Elliptic Instability

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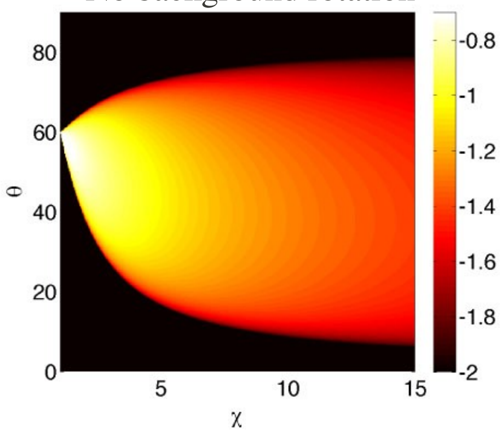

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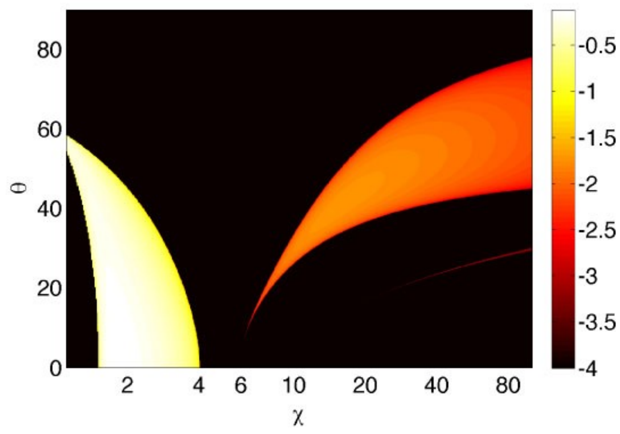

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$$Ro^{-1} < -\frac{b^2}{4}$$

$$\boxed{0 < k < 2|Ro|^{1/2}\frac{\Omega_K}{v_A}}$$

$b = q/Ro$

$q = k/k_{BH}$

$Ro = \dfrac{\Omega_V \delta}{\Omega_K}$

$k_{BH} = \dfrac{\Omega_K}{v_A}$

$\delta = \dfrac{1}{2}(\chi + \chi^{-1})$

Magneto-Elliptic Instability – No vortex limit

$$\boxed{0 < k/k_{BH} < \sqrt{3}}$$

Magneto-Elliptic Instability – No vortex limit

$$\boxed{0 < k / k_{BH} < \sqrt{3}}$$

MRI

Magneto-Elliptic Instability – No vortex limit

$$\boxed{0 < k/k_{BH} < \sqrt{3}}$$

“This spells MRI in huge neon letters for the seasoned disk modeller”

Magneto-Elliptic Instability – No vortex limit.

Note on consistency $$Ro = \frac{\Omega_V \delta}{\Omega_K} = \frac{\omega_V}{2\,\Omega_K} - \frac{3}{4}$$

In the no-vortex limit (ω_v=0) , Ro=-3/4

Kida solution $$\Omega_V = -\frac{3\,\Omega_K}{2(\chi-1)} \longrightarrow Ro = -\frac{3}{4}\frac{\chi^2-1}{\chi(\chi-1)}$$

$$\boxed{\lim_{\chi\to\infty} Ro = -\frac{3}{4}}$$

$$u_x = -\Omega_V\, y/\chi$$
$$u_y = \Omega_V\, x\,\chi$$

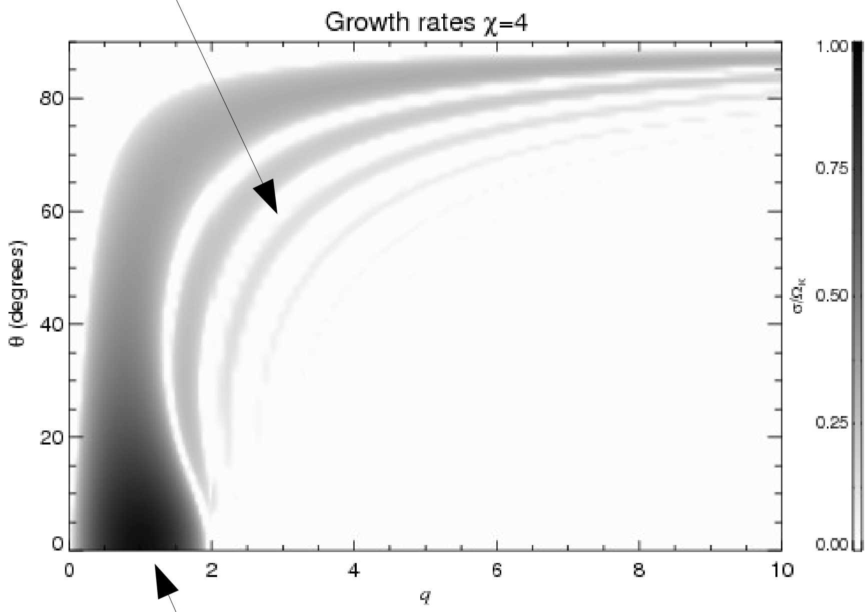
$$\longrightarrow$$

$$\boxed{\begin{aligned} u_x &= 0 \\ u_y &= -3/2\,\Omega_K\, x \end{aligned}}$$

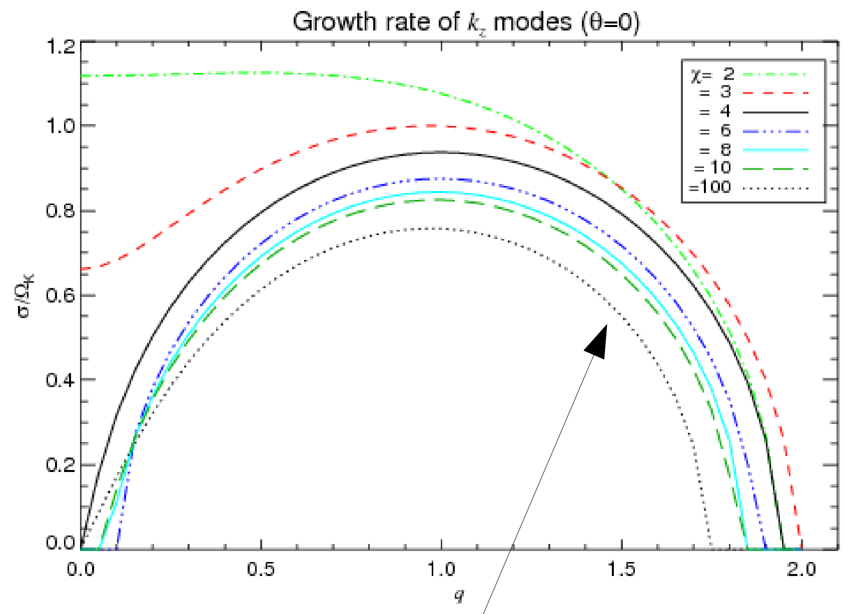
A vortex of infinite aspect ratio is equivalent to a shear flow

Growth rates

Vortex turnover resonance
with Alfvén waves

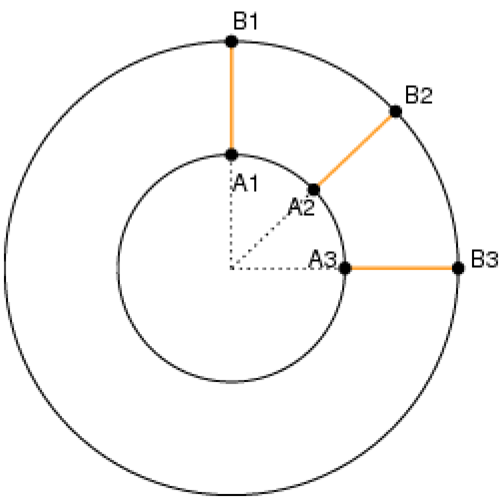


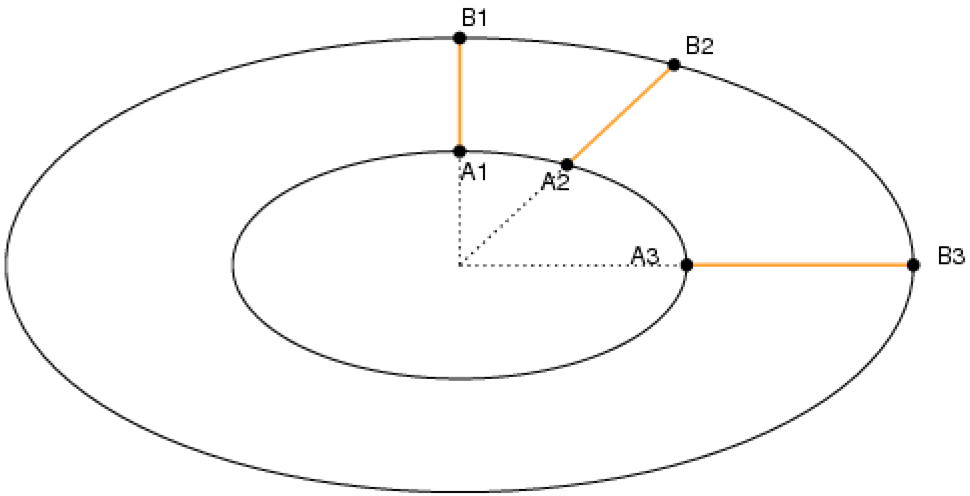
“Horizontal” magneto-elliptic
instability



MRI

Common ground between MRI and MEI



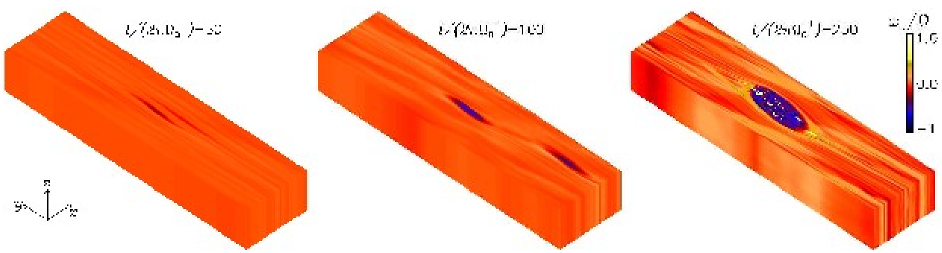


Elliptic streamlines have shear
even in uniform rotation

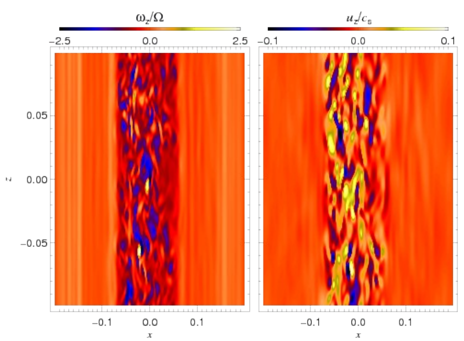
Conclusions

- 3D non-magnetized vortices reach a steady state
  * Unstable yet coherent
  * Balance between baroclinicity (+) and stretching (-)
  * Subsonic core turbulence (10% of sound speed)

- Vortices do not survive the MRI
  * Channel flows
  * Violent core turbulence
    - Magneto-elliptic instability
    - MRI is a limit of the MEI



- Fits neatly in the layered accretion paradigm.
  * Active layers are unmodified
  * Dead zone only is endowed with vortices



Open questions:
- Vertical stratification
- Realistic entropy gradients and thermal diffusion
- *Particles??*

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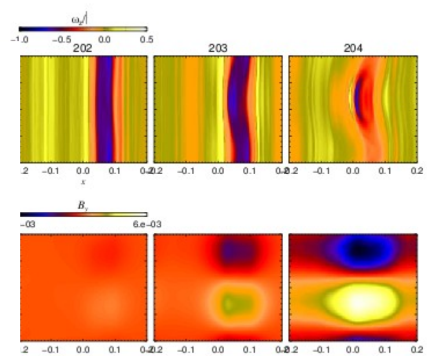

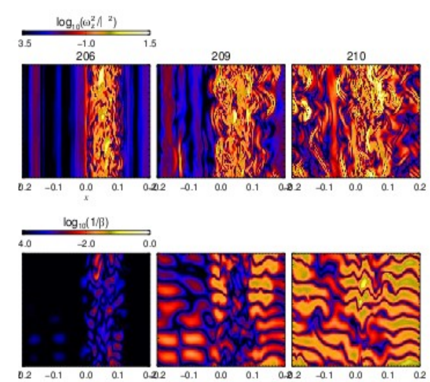

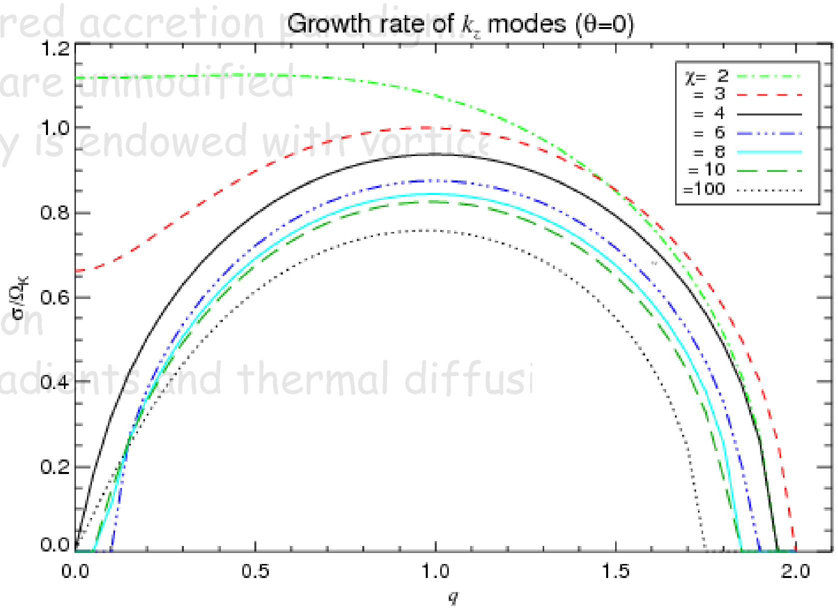

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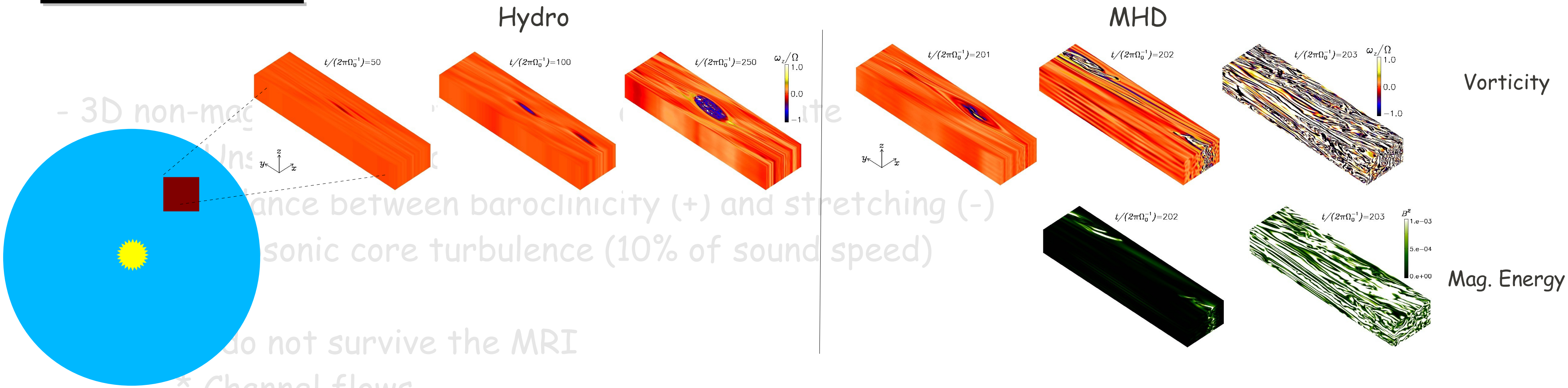

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Theoretical Modelling

Gas

Continuity eq. (density)

$$\frac{\partial \rho_g}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\rho_g - \rho_g \nabla\cdot\boldsymbol{u}$$

Navier-Stokes eq. (momentum)

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\Phi - \rho_g^{-1}\left(\nabla p + \boldsymbol{J}\times\boldsymbol{B} + \rho_p f_d\right)$$

Induction eq. (magnetic field)

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times\left(\boldsymbol{u}\times\boldsymbol{B} - \eta\mu_0\boldsymbol{J}\right)$$

Entropy equation

$$\frac{\partial s}{\partial t} = -(\boldsymbol{u}\cdot\nabla)s + \frac{1}{\rho_g T}\left(\nabla\cdot(K\nabla T) - \rho_g c_V \frac{(T-T_0)}{\tau_c} + \eta\mu_0\boldsymbol{J}^2\right)$$

Eq. of state (pressure)

$$p = \rho_g c_s^2/\gamma$$

Gravitational potential

$$\Phi = \Phi_{sg} - \frac{GM}{r}$$

Poisson equation

$$\nabla^2\Phi_{sg} = 4\pi G(\rho_g + \rho_p)$$

Solids

Drag force

$$f_d = -\left(\frac{3\rho_g C_D|\boldsymbol{w}-\boldsymbol{u}|}{8 a_\bullet \rho_\bullet}\right)(\boldsymbol{w}-\boldsymbol{u})$$

Momentum

$$\frac{d\boldsymbol{w}}{dt} = -\nabla\Phi - f_d$$

Position

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{w}$$