Class 8

Nucleosynthesis

Stellar structure (Cont.)

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$\frac{dP_r}{dr} = -\frac{GM_r}{r^2}\rho$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$$

$$\frac{dT_r}{dr} = \begin{cases} -\frac{3}{16ac}\frac{\kappa_R}{T^3}\frac{L_r}{4\pi r^2} & \text{if } \nabla < \nabla_{ad} \\ \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} & \text{if } \nabla \geq \nabla_{ad} \end{cases}$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

Crude approximation

$$\frac{P}{R} \propto \frac{M}{R^2}\rho \rightarrow \rho \propto \frac{M}{R^3} \rightarrow \boxed{P \propto \frac{M}{R}\rho \propto \frac{M^2}{R^4}} \quad (I)$$

But from $P \propto \rho T$ (eq of state):

$$P \propto \frac{M}{R^3} T \quad (II)$$

To have I and II hold:

$$\frac{M^2}{R^4} \propto \frac{M T}{R^3} \Rightarrow \boxed{T \propto \frac{M}{R}} \quad III$$

Do the same for the temperature equation

$$\frac{dT}{dr} = -\frac{3}{4a_B c}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

$$\frac{T}{R} \propto \frac{M}{R^3 T^3}\frac{L}{R^2} \quad \therefore \quad L \propto \frac{(TR)^4}{M} \qquad (\text{II})$$

But from III, $TR \propto M$

So, $L \propto M^3$ (show figure)

$$t_{nuc} \sim \frac{M}{L} \propto \frac{M}{M^3} \propto M^{-2}$$

A more massive ★ has more nuclear fuel, but it burns at a faster rate.

Also $L = 4\pi R^2 \sigma T^4$ so $L \propto R^2 T^4$

$L \propto M^3$

$RT \propto M$

$$L \propto (RT)^2 T^2 = M^2 T^2$$

So $M \propto T^2$

So $L \propto T^6$ (Main Sequence)

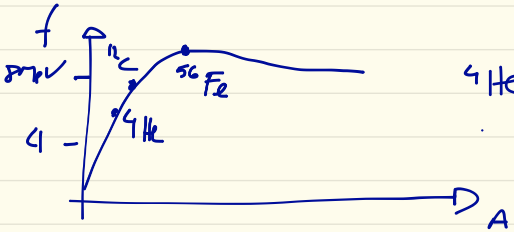
Nucleosynthesys

A nucleus is always found to be less massive than the combined mass of protons and neutrons. The difference is the binding energy

Atomic mass : A
Atomic number : Z
Z protons, A-Z neutrons

$$E_B = \left[Z m_p + (A-Z) m_n - m_{nuc} \right] c^2$$

$$f = \frac{E_B}{A}$$



$^4He \Rightarrow$ 6.6 MeV (0.7% m_p)

Coulomb Barrier

$$V = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e^2}{r}$$

for the Sun, $T = 10^7$ K, typical $k_B T$ energy is of the order of keV. Too low to allow for fusion.

Tunelling has to be taken into account.

Reaction per unit time between nucleus $Z_1 e$ and $Z_2 e$, with volume densities n_1 and n_2:

$$f(v)\,dv = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) 4\pi v^2 dv$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$f(E)\,dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(k_B T)^{3/2}} \exp\left(-\frac{E}{k_B T}\right) dE \qquad (I)$$

$\sigma(E) \Rightarrow$ Reaction cross-section between two nuclei approaching each other with energy E.

Reaction rate per unit volume: $\quad r = n_1 n_2 \langle \sigma v \rangle$

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E)\, v\, f(E)\, dE \qquad (II)$$

Since the velocities are Maxwellian, we only need to know the reaction cross-section $\sigma(E)$ to calculate the reaction rates.

The kinetic energy is much less than the Coulomb barrier. It thus has to depend on the probability of tunnelling through the barrier. (Derive the probability as homework)

$$P \propto \exp\left[-\frac{1}{2\varepsilon_0 h}\left(\frac{m}{2}\right)^{1/2}\frac{Z_1 Z_2 e^2}{\sqrt{E}}\right]$$

Without tunneling, the cross section should be proportional to λ^2, λ = de Broglie wavelength.

$$\lambda = h/p = \frac{h}{mv} = \frac{h}{\sqrt{\frac{m}{2}}v \cdot \sqrt{2}\sqrt{m}}$$

$$= \frac{h}{\sqrt{2mE}} \qquad \therefore \lambda^2 \propto \frac{1}{E}$$

The prob without tunelling: $P \propto \lambda^2 \propto \frac{1}{E}$

Prob with tunneling: $P \propto \exp\left(\frac{-b}{\sqrt{E}}\right)$

$$b = \frac{1}{2\varepsilon_0 h}\left(\frac{m}{2}\right)^{1/2} Z_1 Z_2 e^2$$

So,

$$P \propto \frac{1}{E} \exp\left(-\frac{b}{\sqrt{E}}\right)$$

$\sigma(E) \propto P$

$$\sigma(E) = \frac{S(E)}{E} \exp\left(\frac{-b}{\sqrt{E}}\right) \quad (\text{III})$$

$S(E) \Rightarrow$ Not a constant. Experiments show it is a slow function of E.

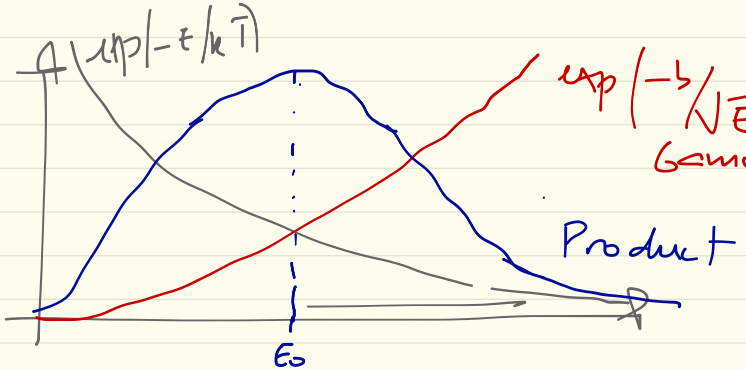
(Slow: weakly dependant)

Occasionally $S(E)$ can spike: resonances

Substitute (III) and (I) into (II)

$$\langle \sigma r \rangle = \frac{2^{3/2}}{\sqrt{\pi m}} \frac{1}{(k_B T)^{3/2}} \int_0^\infty S(E) e^{-\frac{E}{k_B T}} e^{-b/\sqrt{E}} \, dE$$

Maxwell factor



Appreciable value around E_0. Replace $S(E)$ by $S(E_0)$ (constant), and then

$$\langle \sigma r \rangle = \frac{2^{3/2}}{\sqrt{\pi m}} \frac{1}{(K_B T)^{3/2}} \int_0^\infty S(E) e^{-\frac{E}{K_B T}} e^{-b/\sqrt{E}} dE$$

$$\langle \sigma r \rangle = C \int_0^\infty e^{g(E)} dE \quad ; \quad g(E) = -\frac{E}{K_B T} - \frac{b}{\sqrt{E}}$$

(Maxwell) (Gamov)

Find E_0 by $\frac{dg}{dE} = 0$

$$E_0 = \left(\frac{1}{2} b K_B T \right)^{2/3}$$

$$= \left[\left(\frac{m}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2 K_B T}{4\epsilon_0 \hbar} \right]^{2/3}$$

Let $g(E_0) = -\Gamma$

$$\Gamma = -g(E_0) = ?$$

$$E_0 = \left(\frac{1}{2} b K_B T\right)^{2/3}$$

$$-g(E_0) = \frac{E_0}{K_B T} + \frac{b}{\sqrt{E_0}}$$

(Show)

$$r = -g(E_0) = 3\left[\left(\frac{m}{2K_B T}\right)^{1/2} \frac{z_1 z_2 e^2}{4\epsilon_0 \hbar}\right]^{2/3}$$

Expand $g(E)$ around E_0

$$g(E) = g(E_0) + \frac{dg}{dE}(E - E_0) + \frac{1}{2!}\frac{d^2 g}{dE^2}(E - E_0)^2 + \ldots$$

$$\frac{d^2 g}{dE^2} \rightarrow J = e^{-r}\int_0^{\infty} e^{-\frac{r}{4}\left(\frac{E}{E_0} - 1\right)^2} dE$$

↳ Replace by -infinity !

$$J = e^{-r}\int_{-\infty}^{\infty} (\;) dE$$

$$\therefore \quad J = \frac{2}{3} K_B T \sqrt{\pi r}\, e^{-r}$$

$$\langle \sigma v \rangle \propto \frac{S(E_0)}{T^{2/3}} \exp\left(-3\left(\frac{e^4 m z_1 z_2}{32 \epsilon_0^2 K_B \hbar^2 T}\right)^{1/3}\right)$$

r goes as $T^{-1/3}$

$$r \Delta \mathcal{E} = p\mathcal{E} = n_1 n_2 \langle \sigma v \rangle \Delta \mathcal{E}$$

Energy generation per unit volume

$$\varepsilon = C\rho X_1 X_2 \frac{1}{T^{2/3}} \exp\left[-3\left(\frac{e^4}{32\,\varepsilon_0^2 k_B^2}\,\frac{m Z_1^2 Z_2^2}{T}\right)^{1/3}\right]$$

X_1 X_2 (mass fractions of nuclei)

Thus $n_1 \propto \rho X_1$; $n_2 \propto \rho X_2$

- Increases sharply with temperature
- Heavier nuclei ⇒ less likely than with lighter nuclei

Deuterium binding energy : 2.22 MeV
Helium binding energy : 6.6 MeV

Strong force potential : 30 MeV

Equate $K = V$ Kinetic = potential of barrier

$\frac{mv^2}{2} = \frac{3}{2} kT$ for $T \sim 10^7$ K, order of keV.

$$\frac{3}{2} kT = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r} Z_1 Z_2 \approx 10^{10}\,\text{K}$$

Proton must tunnel!

Rewrite $E = p^2/2m$ of order of
Closest approach ✓ de Broglie wavelength.

$T \sim 10^7 K$.

Reaction chains. Must be pair-wise,
Conserve: Electric charge
Baryon number
Lepton number

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$\nabla^2\psi = \frac{2m}{\hbar^2}(V-E)\psi$$

$$\frac{1}{\psi}\frac{d^2\psi}{dr^2} = \frac{2m}{\hbar^2}(V-E)$$

$$\psi = A\exp C$$

$$\psi' = AC'\exp C$$

$$\psi'' = AC''\exp C + A(C')^2\exp C$$

$$\Rightarrow \frac{\psi''}{\psi} = \frac{AC'^2\exp C}{A\exp C}$$

$$C'^2 = \frac{2m}{\hbar^2}(V-E)$$

$$C' = \left[\frac{2m}{\hbar^2}(V-E)\right]^{1/2}$$

$$C = \int\left(\frac{2m}{\hbar^2}(V-E)\right)^{1/2} dr$$

$$P = \exp\left\{-2\int_{r_0}^{r_1}\left[\frac{2m}{\hbar^2}(V-E)\right]^{1/2} dr\right\}$$

Substitute $r = r_1 \cos^2\theta$ and $r_1/r_0 \gg 1$

$$P \propto \exp\left[-\frac{1}{2\varepsilon_0 \hbar}\left(\frac{m}{2}\right)^{1/2} \frac{z_1 z_2 e^2}{\sqrt{E}}\right]$$

Important reactions.

Proton–proton chain

$$^1H + {^1H} \rightarrow {^2H} + e^+ + \nu$$
$$^2H + {^1H} \rightarrow {^3He} + \gamma$$

$$^3He + {^3He} \rightarrow {^4He} + {^1H} + {^1H}$$

Find rate. 1st reaction is mediated by weak force. slow reaction, determines rate. Add all energies, divide by time rate of slowest.

$$\varepsilon_{pp} = 2.4 \times 10^{-11} \rho X^2 \left(\frac{10^6}{T}\right)^{2/3} \exp\left[-33.8\left(\frac{10^6}{T}\right)^{1/3}\right] \frac{\text{erg}}{\text{s g}}$$

(Notice same form as ε)

$S(E)$ measured in lab in MeV, where Coulomb barrier can be neglected, and extrapolated to keV.

other reaction: CNO

$$^{12}C + {}^{1}H \to {}^{13}N + \gamma$$

$$^{13}N \to {}^{13}C + e^{+} + \nu$$

$$^{13}C + {}^{1}H \to {}^{14}N \to \gamma$$

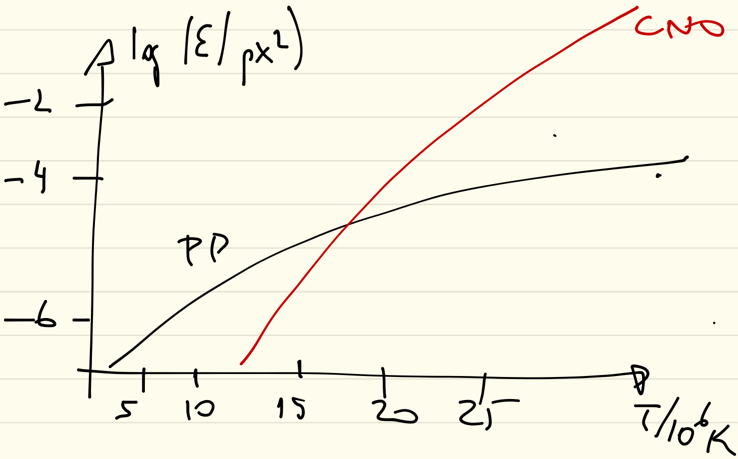
$$^{14}N + {}^{1}H \to {}^{15}O + \gamma \quad \text{(slowest)}$$

$$^{15}O \to {}^{15}N + e^{+} + \nu$$

$$^{15}N + {}^{1}H \to {}^{12}C + {}^{4}He$$

$$\varepsilon_{CNO} = 8.7 \times 10^{10} \rho X_{CNO} X \left(\frac{10^6}{T}\right)^{2/3} \exp\left[-152.3\left(\frac{10^6}{T}\right)^{1/3}\right] \quad \frac{\text{erg}}{\text{s} \cdot \text{g}}$$

$$X_{CNO} = X_C + X_N + X_O$$



PP dominates up to ~ 15×10^7 K

CNO takes over after

* surplus of Be, that can fuse with He.

Triple Alpha

$^1H + {}^4He = {}^5X$ (?)

$^4He + {}^4He = {}^8X$ (?)

$^4He + {}^4He \to {}^8Be + \gamma$ -93.7 keV
$^8Be + {}^4He \to {}^{12}C + \gamma$ $+7.367$ MeV

Nuclide table: No stable nucleus of mass 5 or 8.

Salpeter suggested triple alpha:

$$^4He + {}^4He + {}^4He \to {}^{12}C + \gamma$$

3 particles! Much less likely to occur than 2 particles only.

Also the repulsion is much stronger.

Highly improbable in the early universe, so Big Bang nucleosynthesys should not have gone beyond helium.

In stars, can happen when the temperature goes beyond 10^8 K. (Helium must fuse faster than Be decays)*

Even then, still very unlikely. Too slow if the cross-section was non-resonant.

Hoyle (1954) conjectured that there must be a resonance to make the reaction rate appreciable. The resonance was almost immediately found in laboratory experiments.

The triple alpha is unlikely. It takes a long time to occur. Consequence: didn't occur in the Big Bang because within minutes the temperature fell below the critical point for nuclear fusion.

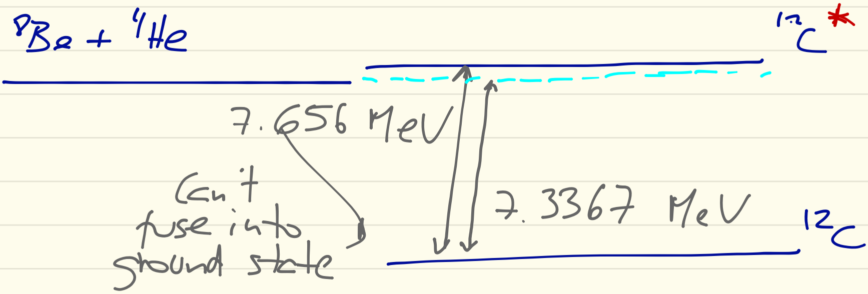
Resonance: ^{8}Be almost exactly same energy as 2 alpha particles:

$$^{4}He + {}^{4}He \rightarrow {}^{8}Be \qquad (100\ keV)$$

$$^{8}Be + {}^{4}He \rightarrow {}^{12}C^{*} \qquad \text{Excited state of carbon}$$

$$^{12}C^{*} \rightarrow {}^{12}C + \gamma \qquad (7.367\ MeV)$$

Gamov: all nucleosynthesis in Big Bang
Hoyle: stellar nucleosynthesis. Stuff formed from hydrogen, the "true" primordial substance.



$^8Be + ^4He$ can use the ~keV energy of the collision to fuse into the excited state, which then transitions to the ground state.

Also: $^{12}C + ^4He \to ^{16}O$ has no resonance. If this existed, All carbon would become oxygen, and C would be as rare as beryllium.

$^{56}Ni + He \rightarrow {}^{60}Zn$ Energy is consumed and the ~~*~~ collapses.

"Iron Peak"

Resonance: transition in the nucleus. Effectively a "nuclear spectral line". It is a "resonance" between the energy of the incoming collision, and differences in the energy levels of the resulting nucleus.

$$\varepsilon_{3\alpha} = 50.9 \times 10^{-10} \rho^2 Y^3 \left(\frac{10^8}{T}\right)^3 \exp\left[-44\left(\frac{10^8}{T}\right)\right] \frac{erg}{s \cdot g}$$