Class 7

Convection

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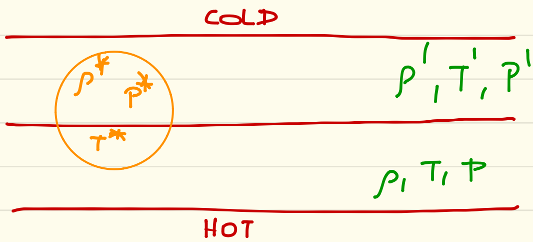
Let us consider a slab of the interior of the star, hotter below and colder above, and in this slab, the upward displacement of a gas bubble.



We will use prime superscripts for the cold slab, plain for the hot, and star superscript for the bubble, as shown aside.

The bubble expands adiabatically:

$$\rho^* = \rho \left(\frac{P^*}{P}\right)^{1/\gamma}$$

The surrounding pressure and density are

$$P' = P + \frac{dP}{dr}\Delta r \qquad \rho' = \rho + \frac{d\rho}{dr}\Delta r$$

In addition, the bubble is always in pressure equilibrium,

$$P^* = P'$$

So, $\rho^* = \rho\left(\frac{P'}{P}\right)^{1/\gamma} = \rho\left(1 + \frac{1}{P}\frac{dP}{dr}\Delta r\right)^{1/\gamma}$

$$= \rho\left(1 + \frac{1}{P\gamma}\frac{dP}{dr}\Delta r\right)$$

Recall now the equation of state

$$P = \rho R T \qquad d\ln P = d\ln \rho + d\ln T$$

$$\therefore \quad d\ln \rho = d\ln P - d\ln T$$

Divide this by dr

$$\frac{d\rho}{dr} = \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}$$

And substitute in the equation for the surrounding density ρ'

$$\rho' = \rho + \left(\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}\right)\Delta r$$

Convection will set in if the bubble keeps rising. That is, is $\rho^* < \rho'$.

$$\rho^* - \rho' = \cancel{\rho} + \frac{\rho}{\gamma P}\frac{dP}{dr}\Delta r$$

$$- \cancel{\rho} - \left(\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}\right)\Delta r$$

$$\rho' = \rho + \left(\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}\right)\Delta r$$ SURROUNDING

$$\rho^* = \rho + \frac{\rho}{\gamma P}\frac{dP}{dr}\Delta r$$ BUBBLE

Condition for buoyancy:

$$\delta\rho = \rho^* - \rho' =$$

$$= \cancel{\rho} + \left(\frac{\rho}{\gamma P}\frac{dP}{dr}\right)\Delta r - \cancel{\rho} - \left(\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}\right)\Delta r$$

$$= \left(-\frac{\rho}{P}\left(1-\frac{1}{\gamma}\right)\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}\right)\Delta r$$

Instability Condition: $\rho^* < \rho'$

(Bubble less dense than surroundings)

$$-\frac{\rho}{P}\left(1-\frac{1}{\gamma}\right)\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr} < 0$$

$$\frac{dT}{dr} < \frac{T}{\cancel{\rho}}\frac{\cancel{\rho}}{P}\left(1-\frac{1}{\gamma}\right)\frac{dP}{dr}$$

$$\frac{dT}{dr} < \frac{T}{\cancel{\rho}} \frac{\cancel{\rho}}{P}\left(1-\frac{1}{\gamma}\right)\frac{dP}{dr}$$

$$\frac{dT}{dr} < \frac{T}{P}\left(1-\frac{1}{\gamma}\right)\frac{dP}{dr}$$

As these quantities are negative

$$\boxed{\left|\frac{dT}{dr}\right| > \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|}$$ Schwarschild stability condition

This defines the adiabatic Temperature gradient

$$\left.\frac{dT}{dr}\right|_{ad} \equiv \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

And the condition for convection is that the gradient is steeper than this

$$\boxed{\left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{ad}}$$ Convection

We can also write

$$\left|\frac{d\ln T}{d\ln r}\right|_{ad} = \left(1-\frac{1}{\gamma}\right)\left|\frac{d\ln P}{d\ln r}\right|$$

A simpler way to write this is

$$\left(\frac{d\ln T}{d\ln P}\right)_{ad} = \left(1-\frac{1}{\gamma}\right)$$

Some author like to call this ∇_{ad}; so

$$\nabla_{ad} \equiv \left(\frac{d\ln T}{d\ln P}\right)_{ad} = 1-\frac{1}{\gamma}$$

Removes the having to keep track of signs because dT and dP have the same sign

So, the condition for convection is

$$\nabla > \nabla_{ad}$$

We can also write in general

$$\nabla \equiv \frac{d\ln T}{d\ln P}$$

The notation, though confusing, is compact and widespread. The radiative gradient is

$$\nabla_{rad} \equiv \left.\frac{d\ln T}{d\ln P}\right|_{rad} = \frac{3 P_\kappa}{16\pi a c G T^4} \frac{L_r}{M_r}$$

with that we have the full equations of stellar structure:

$$\boxed{\begin{aligned}
\frac{dM_r}{dr} &= 4\pi r^2 \rho \\
\frac{dP_r}{dr} &= -\frac{GM_r}{r^2}\rho \\
\frac{dL_r}{dr} &= 4\pi r^2 \rho \varepsilon \\
\frac{dT_r}{dr} &= \begin{cases} -\dfrac{3}{16ac}\dfrac{\kappa_R}{T^3}\dfrac{L_r}{4\pi r^2} & \text{if } \nabla < \nabla_{ad} \\[2ex] \left(1-\dfrac{1}{\gamma}\right)\dfrac{T}{P}\dfrac{dP}{dr} & \text{if } \nabla \geq \nabla_{ad} \end{cases}
\end{aligned}}$$

Hopf function

why 2/3

Chandrasekhar writes

$$J(\tau) = \frac{3F_0}{4\pi}\left(\tau + q(\tau)\right)$$

$q(\tau)$ is called Hopf function

Solving generally for $q(\tau)$ is a tricky business, and essentially a math problem, the values are

0.577 for $\tau = 0$
0.710 for $\tau = +\infty$

Eddington $q \simeq 0.667$

surface temperature:

$T(\tau=0) = T_0$ $\qquad$ $T_0^4 = \frac{3}{4} T_{eff}^4 \cdot q(\tau)$

Grey:

$T_0 = \sqrt[4]{1/2}\, T_{eff} = 0.841\, T_{eff} = 4860\,K$

Chandra: $\quad T_0 = 0.811\, T_{eff} = 4690\,K$

∇T of grey atmosphere and

1 photon mean free path

opacity $K_{5000Å} \simeq 0.03\ m^2 Kg^{-1}$; $\rho \simeq 2 \cdot 10^{-7} \frac{g}{cm^3}$

$$\bar{\ell} = \frac{1}{K_{5000Å}\,\rho} \simeq 160\,Km \quad \text{(about 100 miles)}$$

Temperature scale height:

$$H_T = \frac{T}{\left|\frac{dT}{dr}\right|} = \frac{T}{K\rho \frac{dT}{d\tau}}$$

$$T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + 2/3\right)$$

$$\frac{d(T^4)}{d\tau} = \frac{3}{4} T_{eff}^4 = 4T^3 \frac{dT}{d\tau}$$

$$\therefore \frac{dT}{d\tau} = \frac{3}{16T^3} T_{eff}^4$$

$$H_T = \frac{16\,T^4}{3K\rho\,T_{eff}^4} = \frac{\cancel{16}^{4}}{\cancel{3}}\,\bar{\ell} \cdot \frac{\cancel{3}}{\cancel{4}} \left(\tau + 2/3\right)$$

At $\tau = 0$; $H_T = \frac{8\bar{\ell}}{3} \simeq 400\ km$

The mean free path is comparable to the scale at which temperature changes occurs. As a result, photons produced at different temperatures reach the photosphere.