Class notes 6 - (Sept 14th, 2015)

1 Stellar Structure

We have derived in last class two of the equations that determine the interior structure of a star. The mass continuity equation

$$\frac{dM}{dr} = 4\pi r^2 \rho dr \tag{1}$$

And the hydrostatic equilibrium equation

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \tag{2}$$

Let us now derive the other equations of the set.

2 Stellar energy

A star is a gravitationally bound object. Its total gravitational energy is defined as the negative of the energy required to separate its constituent particles to infinity. Let us calculate this energy. Suppose we have a spherical shell of mass dM. To move this shell from radius r to $r + dr$ requires $F \cdot dr$ units of work. The force F is the gravitational force, so the energy $d\Omega$ needed to move it from r to infinity is

$$\begin{aligned} d\Omega &= -\int_r^\infty \frac{GM_r}{r'^2} dM dr' && (3) \\ &= -\frac{GM_r}{r} dM_r && (4) \end{aligned}$$

To disperse the whole star requires that we do that for all dM, or

$$\Omega = \int d\Omega = -\int_0^M \frac{GM_r}{r} dM_r \tag{5}$$

We can write that as $\Omega = -qGM^2/R$, where q is of order unity and reflects the mass distribution within the star. For constant density,

$$\begin{aligned} dM_r &= 4\pi r^2 \rho dr && (6) \\ M_r &= \frac{r^3}{R^3} M && (7) \end{aligned}$$

Then

$$\begin{aligned} \Omega &= -\frac{4\pi G \rho M}{R^3} \int_0^R r^4 dr && (8) \\ &= -\frac{4\pi}{5} G \rho M R^2 && (9) \end{aligned}$$

And because $\rho = 3M/(4\pi R^3)$, we have

$$\Omega = -\frac{3}{5}\frac{GM^2}{R} \tag{10}$$

i.e, $q = 3/5$ for constant density.

3 Virial Theorem

Consider the scalar $\sum_i p_i r_i$. Derive that in time so that

$$\frac{d}{dt}\sum_i p_i r_i = \sum_i \dot{p}_i r_i + p_i \dot{r}_i \tag{11}$$

The last term is $\sum_i m_i v_i^2$, equal to twice the total kinetic energy of the star, $2K$. Recalling also that $\dot{p}_i = F_i$, the first term is $\sum_i \dot{F}_i r_i$, or the work done by the gravitational force. This is the quantity Ω derived first. So,

$$2K + \Omega = \frac{d}{dt}\sum_i p_i r_i \tag{12}$$

We can evaluate $\frac{d}{dt}\sum_i p_i r_i$ by noticing this is

$$\frac{d}{dt}\sum_i m_i \dot{r}_i r_i = \frac{1}{2}\frac{d}{dt}\sum_i \frac{d}{dt}(m_i r_i^2) = \frac{1}{2}\frac{d^2 I}{dt^2}$$

where I is the inertia moment. If $d^2I/dt^2 = 0$,

$$2K + \Omega = 0 \tag{13}$$

The virial theorem. According to the virial theorem, the total energy is half the gravitational potential energy

$$E = K + \Omega = \frac{1}{2}\Omega = -{}^1\!/_2|\Omega| \tag{14}$$

Notice also that as a star steadily contracts (so that $d^2I/dt^2 = 0$), it becomes more gravitationally bound, making Ω larger, and it follows that K also becomes larger, so the star becomes hotter. By how much? Suppose the contracting star increases its boundness by $\Delta\Omega$. According to Eq. (13), the change in temperature is

$$2(K + \Delta K) + (\Omega + \Delta\Omega) = 0 \tag{15}$$

$$\Delta K = -\frac{1}{2}\Delta\Omega = \frac{1}{2}|\Delta\Omega| \tag{16}$$

The total energy is

$$
\begin{aligned}
E &= (K+\Delta K)+(\Omega+\Delta\Omega) && (17)\\
&= E_0+(\Delta K+\Delta\Omega) && (18)\\
&= E_0+(-{}^1/_2\Delta\Omega+\Delta\Omega) && (19)\\
&= E_0+{}^1/_2\Delta\Omega && (20)\\
&= E_0-{}^1/_2|\Delta\Omega| && (21)
\end{aligned}
$$

Yet the energy must be conserved. The energy of the star was decreased by $-{}^1/_2|\Delta\Omega|$. Where did this energy go? It must have left the system as radiation. This leads to a very elegant result: as a star contracts, half of the energy removed from the gravitational field remains in the system as kinetic energy (increasing the temperature) and half is radiated away. *Gravitational contraction leads to release of radiation*. Helmholtz and Kelvin suggested that this is how stars shine. For the sun, $GM_\odot/R_\odot \approx 3.8\times10^{48}$ ergs. Dividing it by the solar luminosity, we get a timescale of about 10^7 yrs, which cannot be the whole story (Kelvin thought that it was, and never accepted that the Earth could be older than that, in spite of fossil evidence).

4 Luminosity equation

We can derive an equation for the luminosity also in terms of elementary considerations. Define the energy production rate ε as energy per unit time per mass per unit volume. In a spherical shell the luminosity is thus

$$dL_r = 4\pi r^2 dr\rho\varepsilon \tag{22}$$

or

$$\frac{dL_r}{dr} = 4\pi r^2\rho\varepsilon \tag{23}$$

5 Temperature equation

Knowing the luminosity,

$$L_r = 4\pi r^2 F_r \tag{24}$$

the condition of radiative equilibrium gives an equation for the temperature. In here, F_r is given by the condition of radiative equilibrium $\boldsymbol{\nabla}\cdot\boldsymbol{F} = \partial_r F = 0$, which yields

$$F_r = -\frac{16}{3}\frac{\sigma T^3}{\kappa_R}\frac{dT}{dr} \tag{25}$$

So, we can write

$$\frac{dT}{dr} = -\frac{3\kappa_R L_r}{16\pi r^2 c a_R T^3} \tag{26}$$

where we substituted $a_R = \sigma/(4c)$