Class notes 5 - (Sept 9th, 2015)

1 Pressure and flux in radiative equilibrium

In radiative equilibrium, the condition for pressure can be found by multiplying the RT equation by $\cos\theta$, and integrating in angle.

$$\frac{d}{d\tau_\nu}\oint I_\nu \cos^2\theta d\omega = \oint I_\nu \cos\theta d\omega - S_\nu \oint \cos\theta d\omega \tag{1}$$

The first integral is $4\pi K_\nu$, the second the flux, and the latter zero. So,

$$\frac{dK_\nu}{d\tau_\nu} = H_\nu \tag{2}$$

Integrating it in frequency,

$$\int_0^\infty \frac{dK_\nu}{d\tau_\nu} d\nu = \frac{1}{4\pi}\int_0^\infty F_\nu d\nu = \frac{F_0}{4\pi} = H_0 \tag{3}$$

2 Gray approximation

The gray approximation assumes that the opacity does not depend on wavelength. So, we simply integrate the RT equation in frequency, to find

$$\mu\frac{dI}{d\tau} = S - I \tag{4}$$

With that approximation, the eqs of radiative equilibrium are

$$F = F_0 \tag{5}$$

$$J = S \tag{6}$$

$$K = \frac{F_0}{4\pi}\tau + \text{const} \tag{7}$$

In the gray atmosphere, the source function is simply the mean intensity.

3 Rossland approximation

Use the plane-parallel approximation

$$\mu\frac{dI_\nu}{dz} = \kappa_\nu\rho(S_\nu - I_\nu) \tag{8}$$

$$I_\nu \approx B_\nu - \frac{\mu}{\kappa_\nu}\frac{\partial B_\nu}{\partial z} \tag{9}$$

$$
\begin{aligned}
F_\nu &= \oint I_\nu \cos\theta d\omega = 2\pi \int_{-1}^{1} I_\nu \mu d\mu & (10)\\
&= -\frac{2\pi}{\kappa_n u}\frac{\partial B_\nu}{\partial z}\int_{-1}^{1}\mu^2 d\mu = -\frac{4\pi}{3\kappa_\nu}\frac{\partial B_\nu}{\partial z} & (11)\\
&= -\frac{4\pi}{3\kappa_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z} & (12)
\end{aligned}
$$

$$
F = \int_0^\infty F_\nu d\nu = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T} d\nu \tag{13}
$$

To bypass the frequency integration, we can define the **Rossland mean opacity**

$$
\frac{1}{\kappa_R} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} \tag{14}
$$

And because

$$
\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T}\int_0^\infty B_\nu d\nu = \frac{\partial \sigma/\pi T^4}{\partial T} = \frac{4\sigma}{\pi}T^3 \tag{15}
$$

So,

$$
F(z) = -\frac{16}{3}\frac{\sigma T^3}{\kappa_R}\frac{\partial T}{\partial z} \tag{16}
$$

Rossland is a very good approximation for the optically thick case, where $\tau_\nu \gg 1$. It is valid whenever the radiation field is isotropic over distances comparable to or less than a radiation mean free path, such as in local thermal equilibrium. The Rossland mean opacity is a weighted average of κ_ν^{-1}, so that frequencies at which the opacity is small tend to dominate the flux: the Rossland mean opacity controls the transport of radiation.

4 Rossland vs Planck mean opacities

The Planck mean opacity is defined as

$$
\kappa_P \equiv \frac{\int_0^\infty \kappa_\nu B_\nu d\nu}{\int_0^\infty B_\nu d\nu} \tag{17}
$$

In constrast to the Rossland mean opacity, that favors transparent wavelengths and thus controls the radiation flux, the Planck opacity is a weighted averaged that favors high opacities, strong absorption lines. So, the Planck opacity is the opacity of choice for describing processes such as absorption and emission.

4.1 The Eddington approximation

An ingeniuous solution to the gray case in the plane-parallel approximation was presented by Eddington. Eddington assumed that the intensity could be decomposed into the contribution from two directions. In this two-ray approximation, we have I_+ as the intensity directed outwards (from the stellar interior toward the surface), and I_- as the intensity directed intwards (from the surface to the interior). That is, the radiation field is

$$I = \begin{cases} I_+ & \text{for} \quad 0 \leq \theta < \pi/2,\ 0 < \phi < 2\pi \\ I_- & \text{for} \quad \pi/2 \leq \theta < \pi,\ 0 < \phi < 2\pi \end{cases} \tag{18}$$

We can then express J, F, and P in terms of this intensity field.

The mean intensity is J is

$$\begin{aligned} J &= \frac{1}{4\pi} \oint I d\omega & (19) \\ &= \frac{1}{4\pi} 2\pi \left(\int_0^{\pi/2} I_+ \sin\theta d\omega + \int_{\pi/2}^{\pi} I_- \sin\theta d\omega \right) & (20) \\ &= \frac{1}{2} \left(I_+ + I_- \right) & (21) \end{aligned}$$

The flux is

$$\begin{aligned} F &= \oint I \cos\theta d\omega & (22) \\ &= 2\pi \left(\int_0^{\pi/2} I_+ \cos\theta \sin\theta d\omega + \int_{\pi/2}^{\pi} I_- \cos\theta \sin\theta d\omega \right) & (23) \\ &= \pi \left(I_+ - I_- \right) & (24) \end{aligned}$$

And the quantity K is

$$\begin{aligned} K &= \frac{1}{4\pi} \oint I \cos^2\theta d\omega & (25) \\ &= \frac{1}{6} \left(I_+ + I_- \right) = \frac{J}{3} & (26) \end{aligned}$$

That is,

$$P = \frac{4\pi}{3c} J \tag{27}$$

And, according to radiative equilibrium,

$$\frac{4\pi}{3c} J = \frac{F\tau}{c} + C \tag{28}$$

We can evaluate the constant of integration C by the boundary condition, that at the surface $\tau = 0$ and $I_- = 0$. So, at the upper layer $J(\tau = 0) = F/(2\pi)$, and thus $C = 2F/(3c)$. So,

$$\frac{4\pi}{3} J = F \left(\tau + \frac{2}{3} \right) \tag{29}$$

And, since $F = \sigma T_{\text{eff}}^4$

$$J = \frac{3\sigma}{4\pi} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right) \tag{30}$$

In LTE, $S_\nu = B_\nu$, so $S = B = \int_0^\infty B_\nu d\nu = \pi^{-1}\sigma T^4$.

And this must be $S = J$ given the gray approximation. So,$J = \pi^{-1}\sigma T^4$. Plugging that into Eq. (30), we have

$$T^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right) \tag{31}$$

This result shows that the temperature is equal to the effective temperature not at the surface (τ=0), but at the depth where $\tau = 2/3$. Though an approximation, this pertains reasonable well to real stars. This stems from the fact that in the outer atmosphere of a star, the mean free path of a photon is comparable to the length scale of the temperature stratification, so we see not only a single temperature, but layers in a range of temperatures. The depth of $\tau = 2/3$ is the average point of origin of the observed photons.

5 Stellar Structure

We will now derive the equations that determine the interior structure of a star. Consider the amount of mass in a volume element

$$dm = \rho dV = \rho r^2 \sin\theta dr d\theta d\phi \tag{32}$$

So, the amount of mass in a spherically symmetric shell of constant density is

$$dM = \oint_\omega dm = \left(\rho r^2 dr\right)\oint \sin\theta dr d\theta d\phi = 4\pi r^2 \rho dr \tag{33}$$

From that we can write the mass continuity equation

$$\frac{dM}{dr} = 4\pi r^2 \rho dr \tag{34}$$

We can also derive an equation for the force balance. A gas parcel feels a gravity force towards the center of star, given by the mass inside its shell

$$dF_g = -\frac{GM_r dm}{r^2} \tag{35}$$

where $M_r = \int_0^r 4\pi\rho r'^2 dr'$ and m is the mass of the gas parcel. This gravity has to be balanced by the pressure force, the difference in pressure from the base of the gas parcel to its top.

Considering the gas parcel to have area dA, the force above is PdA and the force below is $(P + dP)dA$. The difference is thus

$$dF_p = -dPdA \tag{36}$$

We now consider the forces to be in balance

$$dPdA = -\frac{GM_r dm}{r^2} \tag{37}$$

and considering that $dm = \rho dA dr$,

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \tag{38}$$

6 A constant density star

$$M_r = \frac{4\pi}{3} r^3 \rho = \frac{r^3}{R^3} M \tag{39}$$

$$\frac{dP}{dM_r} = -\frac{GM}{4\pi R^4} \left(\frac{M_r}{M} \right)^{-1/3} \tag{40}$$

Integrate that with a zero pressure boundary

$$\begin{aligned} P &= \int_0^M \frac{dP}{dM_r} dM_r && (41) \\ &= P_c \left[1 - \left(\frac{M_r}{M}^{2/3} \right) \right] && (42) \\ &= P_c \left[1 - \left(\frac{r}{R} \right)^2 \right] && (43) \end{aligned}$$

P_c is the central pressure,

$$\begin{aligned} P_c &= \frac{3}{8\pi} \frac{GM^2}{R^4} && (44) \\ &= 1.34 \times 10^{15} \left(\frac{M}{M_\odot} \right)^2 \left(\frac{R}{R_\odot} \right)^{-4} \text{dyne cm}^{-2} && (45) \\ &\approx 10^9 \left(\frac{M}{M_\odot} \right)^2 \left(\frac{R}{R_\odot} \right)^{-4} \text{atm} && (46) \\ & && (47) \end{aligned}$$

Substituting the equation of state, we have the temperature

$$P = c_v \rho T \tag{48}$$

$$T_c = \frac{1}{2} \frac{GM}{R} \frac{\mu}{N_A k} \approx 10^7 \left(\frac{M}{M_\odot} \right) \left(\frac{R}{R_\odot} \right)^{-1} K \tag{49}$$