Class notes - 4 (Sep 2nd, 2015)

1 Microphysics of radiative transfer

Local radiative equilibrium means that all processes are in detailed balance (emission = absorption, excitation = de-excitation, ionization = recombination). No change in the quantities.

Get Kirchhoff 1st law, $S_\nu = B_\nu$ for a blackbody. So, $S_\nu = j_\nu/\kappa_\nu = B_\nu$. That is

$$j_\nu = \kappa_\nu B_\nu \tag{1}$$

This implies some relationship emission and absorption at the microscopic level. Einstein unveiled that relationship. Consider two discrete energy levels (draw them).

Processes:

Spontaneous emission (2-1) : A_{21}, transition probability per unit time for spontaneous emission.

Radiative excitation : B_{12} transition probability per unit time for radiative excitation. Presence of a photon $h\nu$, proportional to density of photons, $B_{12}J$.

Stimulated emission: B_{21} transition probability per unit time for stimulated emission.

In LTE,

$$n_1 B_{12} J = n_2 A_{21} + n_2 B_{21} J \tag{2}$$

n_1 and n_2 are the number densities of atoms in 1 and 2. Isolating J

$$J = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1} \tag{3}$$

In LTE, the ratio n_1/n_2 is given by the Boltzmann relation (see Sect 2.1)

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E_1/KT}}{g_2 e^{-(E_1+h\nu)/KT}} = \frac{g_1}{g_2} e^{h\nu/KT} \tag{4}$$

So,

$$J = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21} e^{h\nu/KT} - 1)} \tag{5}$$

$J \equiv B_\nu$ for black body,

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/KT} - 1} \tag{6}$$

For Eq. (5) and Eq. (6) to be equal at all temperatures, one needs

$$g_1 B_{12} = g_2 B_{21} \tag{7}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{8}$$

These are the *Einstein relations*. Consider the energy absorbed:

$$\kappa_\nu \rho I_\nu = h\nu n_1 B_{12} J_\nu, \tag{9}$$

and the energy emitted:

$$j_\nu \rho = h\nu n_2 B_{21} J_\nu + h\nu n_2 A_{21}. \tag{10}$$

It is seen that the 1st term, being proportional to the intensity, can be treated as negative absorption. So, we can write the effective absorption as

$$\kappa_\nu \rho I_\nu = \frac{h\nu I_\nu}{4\pi}\left(n_1 B_{12} - n_2 B_{21}\right) \tag{11}$$

and the effective emission as

$$j_\nu \rho I_\nu = \frac{h\nu}{4\pi} n_2 A_{12} I_\nu \tag{12}$$

crossing I_ν and using the Einstein relation $B_{21} = g_1/g_2 B_{12}$.

$$\kappa_\nu \rho \quad = \quad \frac{h\nu}{4\pi}\left(n_1 B_{12} - n_2 \frac{g_1}{g_2} B_{12}\right) \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{13}$$

This leads us to the source function in term of the Einstein coefficients:

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{n_2 A_{12}}{n_1 B_{12}(1 - \frac{n_2}{n_1}\frac{g_1}{g_2})} \tag{14}$$

We can identify three behaviors here. The first one is thermal radiation, for which $n_2/n_1 = g_2/g_1 e^{-h\nu/KT}$.

$$S \quad = \quad \frac{A_{21}}{B_{12} g_1/g_2 (e^{h\nu/KT - 1})} \tag{15}$$

$$\quad = \quad \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/KT} - 1} \tag{16}$$

$$\kappa_\nu \rho = \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - e^{-h\nu/KT}\right) \tag{17}$$

Second: Nonthermal radiation.

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} e^{h\nu/KT} \tag{18}$$

Third : Maser.

$$n_1 g_2 < n_2 g_1 \tag{19}$$

If $n_2 \gg n_1$ there are too many atoms in the upper state. The Intensity increases along the ray due to intense stimulated emission. This is called a maser.

2 Scattering

Scattering is any process that changes the direction of a radiation ray. The scatterers in stars are mostly nonrelativistic electrons. For these, scattering is isotropic and coherent. The scattering is also elastic, so the energy aborbed is equal to the energy emitted.

$$\int \kappa_\nu \rho I_\nu d\omega = \int j_\nu \rho d\omega \tag{20}$$

$$j_\nu = \sigma_\nu \rho J_\nu \tag{21}$$

For pure scattering,

$$\frac{dI_\nu}{ds} = -\sigma_\nu \rho (I_\nu - J_\nu) \tag{22}$$

Considering both absorption and scattering

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\kappa_\nu \rho (I_\nu - B_\nu) - \sigma_\nu \rho (I_\nu - J_\nu) && (23) \\ &= -(\kappa_\nu + \sigma_\nu) \rho (I_\nu - S_\nu) && (24) \end{aligned}$$

with the source function being given by

$$S_\nu = \frac{\kappa_\nu B_\nu + \sigma_\nu J_\nu}{\kappa_\nu + \sigma_\nu} \tag{25}$$

$\kappa_\nu + \sigma_\nu$ is the net absorption coefficient, or *extinction* coefficient.

$$d\tau_\nu = (\kappa_\nu + \sigma_\nu) \rho ds \tag{26}$$

2.1 Boltzmann ratios

In LTE, the ratio n_1 and n_2 between the occupancy of excitation levels is given by Boltzmann's law

$$P(E) = g(E) e^{-E/KT} \tag{27}$$

where $P(E)$ is the probability of the system being at the state of energy E, and $g(E)$ is the degeneracy of that level, its statistical weight.

It can be derived by the definition of entropy

$$S = -k \sum_i p_i \ln p_i \tag{28}$$

with the constrains that $\sum_i p_i = 1$ and $< E >= \sum_i p_i E_i = U$ where U is the total energy. The probability p_i is that that maximizes the entropy. We rewrite the equation above using Lagrange multipliers

$$S = -k \sum_i p_i \ln p_i + \lambda_1 \left(\sum_i p_i - 1 \right) + \lambda_2 \left(\sum_i p_i E_i - U \right) \tag{29}$$

We find λ_2 directly by derivating wrt U, and applying the second law of thermodynamics

$$\frac{dS}{dU} = -\lambda_2 = \frac{1}{T} \tag{30}$$

Whereas derivating wrt p_i, which should be zero (probability that maximizes entropy) yields

$$\frac{dS}{dp_i} = 0 = -k \ln p_i - k + \lambda_1 - \frac{E_i}{T} \tag{31}$$

Now isolate p_i

$$p_i = \exp\left[\frac{1}{k}\left(-1 + \lambda_1 - \frac{E_i}{T}\right)\right] = \exp\left[\frac{1}{k}(\lambda_1 - 1)\right] \exp\left(-E_i/kT\right) \tag{32}$$

To find λ_1 we now apply $\sum_i p_i = 1$

$$\exp\left[\frac{1}{k}(\lambda_1 - 1)\right] \sum_i \exp\left(-E_i/kT\right) = 1 \tag{33}$$

that is

$$\lambda_1 = 1 - k \ln Z \tag{34}$$

where

$$Z = \sum_i e^{-E_i/kT} \tag{35}$$

is the partition function. We can also write

$$p_i = \frac{1}{Z} e^{-E_i/KT} \tag{36}$$

2.2 Radiation Pressure

Radiation also carries momentum, so it exerts pressure over a surface. To get the momentum flux, recall that the momentum of a photon is E/c, in the direction of the beam $\hat{\boldsymbol{n}}'$

$$d\boldsymbol{p}_\nu = \frac{dE_\nu}{c}\hat{\boldsymbol{n}}' = \frac{I_\nu}{c} d\boldsymbol{A} \cdot d\boldsymbol{\omega} dt d\nu \hat{\boldsymbol{n}}' = \frac{I_\nu \cos\theta}{c} dA d\omega dt d\nu \hat{\boldsymbol{n}}' \tag{37}$$

The pressure is found by finding the normal of this momentum to the area element, which is normal to $\hat{\boldsymbol{n}}$

$$\begin{aligned} dp_\nu^{\text{perp}} &= d\boldsymbol{p}_\nu \cdot \hat{\boldsymbol{n}} & (38) \\ &= \frac{I_\nu \cos\theta}{c} dA d\omega dt d\nu \hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}} & (39) \\ &= \frac{I_\nu \cos^2\theta}{c} dA d\omega dt d\nu & (40) \end{aligned}$$

And the pressure is the force over area,

$$dP_\nu = \frac{dp^{\mathrm{perp}}}{dtdA} \tag{41}$$
$$= \frac{I_\nu \cos^2\theta}{c} d\omega d\nu \tag{42}$$

Integrating it over all directions, we get the total pressure

$$P_\nu = \frac{1}{c}\oint I_\nu \cos^2\theta d\omega \tag{43}$$

2.3 The quantities H_ν and K_ν

Similarly to the mean intensity, which is the directional average of the intensity, we can divide the flux and pressure by $\oint d\omega$ to write

$$J_\nu = \frac{1}{4\pi}\oint I_\nu d\omega \tag{44}$$
$$H_\nu = \frac{1}{4\pi}\oint I_\nu \cos\theta d\omega \tag{45}$$
$$K_\nu = \frac{1}{4\pi}\oint I_\nu \cos^2\theta d\omega \tag{46}$$

$H_\nu = F_\nu/4\pi$ and $K_\nu = cP_\nu/4\pi$. These are just normalizations to remove factors of 4π that sometimes appear in equations. The interesting fact to notice is that J, H, and K are, respectively, the zeroth, first, and second moments of the intensity with respect to the direction. (moment = combination of a physical quantity and a coordinate).

2.4 Spherical coordinates and the plane-parallel approximation

Let's now find some solutions to the equation of radiative transfer. The optical depth is defined along a ray, whereas in the case of a star, we want to define a direction, towards the observer, that we call the axis z. We can write

$$\frac{d}{dz} = \frac{dr}{dz}\frac{\partial}{\partial r} + \frac{d\theta}{dz}\frac{\partial}{\partial\theta} \tag{47}$$

with $dr = \cos\theta dz$ and $rd\theta = -\sin\theta dz$. The RT equation

$$\frac{\partial I_\nu}{\partial r}\frac{\cos\theta}{\kappa_\nu\rho} - \frac{\partial I_\nu}{\partial\theta}\frac{\sin\theta}{\kappa_\nu\rho r} = S_\nu - I_\nu \tag{48}$$

In the *plane-parallel approximation*, we assume that $r >> 1$, ignoring the curvature. The RT equation then becomes

$$\mu\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \tag{49}$$

where $\mu = \cos\theta$. This is similar to rewriting it in terms of an effective optical depth $\tau_\nu \to \mu^{-1}\tau_\nu$.

2.5 Radiative equilibrium

The atmosphere of a star simplys transports all energy that flows through it, produced in the core. Without sources and sinks, the energy is rigorously conserved. Applying energy conservation to the interior of the star and using Gauss theorem

$$\frac{\partial E}{\partial t} = 0 = \oint \boldsymbol{F} \cdot d\boldsymbol{A} = \int (\boldsymbol{\nabla} \cdot \boldsymbol{F})\, dV \tag{50}$$

The integrand must be zero for all volume elements. So,

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0 \tag{51}$$

For the plane-parallel approximation, that is simply $dF/dx = 0$, or $F \equiv$ const. We can define

$$\int_0^\infty F_\nu d\nu = F_0 = \frac{L}{4\pi r^2} = \sigma T_{\rm eff}^4 \tag{52}$$

Also, in radiative equilibrium, we see that

$$\cos\theta \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \tag{53}$$

integrating in solid angle

$$\frac{d}{d\tau_\nu} \oint I_\nu \cos\theta d\omega = \oint S_\nu d\omega - \oint I_\nu d\omega \tag{54}$$

assuming that S_ν is isotropic (radiative equilibrium)

$$\frac{d}{dz} F_\nu = 4\pi \kappa_\nu \rho (S_\nu - J_\nu) \tag{55}$$

and we integrate that in frequency

$$\frac{d}{dz} \int_0^\infty F_\nu d\nu = 4\pi\rho \int_0^\infty \kappa_\nu (S_\nu - J_\nu) d\nu \tag{56}$$

The LHS is $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, which in radiative equilibrium is zero. Therefore, the RHS is also zero. So, we find the result

$$\int_0^\infty \kappa_\nu S_\nu d\nu = \int_0^\infty \kappa_\nu J_\nu d\nu \tag{57}$$

Or, recalling $S_\nu = j_\nu/\kappa_\nu$,

$$\int_0^\infty j_\nu d\nu = \int_0^\infty \kappa_\nu J_\nu d\nu \tag{58}$$

Bolometric emission = bolometric absorption. What is emitted in a wavelength is absorbed in another. The total energy is conserved.

In radiative equilibrium, the condition for pressure can be found by multiplying the RT equation by $\cos\theta$, and integrating in angle.

$$\frac{d}{d\tau_\nu} \oint I_\nu \cos^2\theta d\omega = \oint I_\nu \cos\theta d\omega - S_\nu \oint \cos\theta d\omega \tag{59}$$

The first integral is $4\pi K_\nu$, the second the flux, and the latter zero. So,

$$\frac{dK_\nu}{d\tau_\nu} = H_\nu \tag{60}$$

Integrating it in frequency,

$$\int_0^\infty \frac{dK_\nu}{d\tau_\nu} d\nu = \frac{1}{4\pi} \int_0^\infty F_\nu d\nu = \frac{F_0}{4\pi} = H_0 \tag{61}$$

2.6 Gray approximation

The gray approximation assumes that the opacity does not depend on wavelength. So, we simply integrate the RT equation in frequency, to find

$$\mu \frac{dI}{d\tau} = S - I \tag{62}$$

With that approximation, the eqs of radiative equilibrium are

$$F = F_0 \tag{63}$$

$$J = S \tag{64}$$

$$K = \frac{F_0}{4\pi}\tau + \text{const} \tag{65}$$

In the gray atmosphere, the source function is simply the mean intensity.