Class notes - 3 (Aug 31st, 2015)

1 Emission and absorption of radiation

1.1 Absorption

The amount of radiation absorbed as a ray traverses a slab of material is proportional to the incoming intensity, and to the path length traversed.

$$dI_\nu = -\alpha_\nu I_\nu ds \tag{1}$$

α_ν is called absorption coefficient, and dI_ν is negative. Frequently we also use

$$dI_\nu = -\kappa_\nu \rho I_\nu ds \tag{2}$$

where the units are $\kappa_\nu\rho \rightarrow \mathrm{cm}^{-1}$ and $\kappa_\nu \rightarrow \mathrm{cm}^2\,\mathrm{g}^{-1}$.

κ_ν is usually called mass absorption coefficient, or *opacity*.

The absorption law is phenomenological. Imagine an ensemble of absorbers of number density n, all having cross section σ_ν. The total number of absorbers in a volume $dV = dA'ds$ is $N = ndA'ds$, and collectively they offer an absorbing area $ndA'ds\sigma_\nu$. So, the amount of energy absorbed from the beam is

$$dE_\nu = -dI_\nu dA d\omega dt d\nu \propto (ndA'ds\sigma_\nu) d\omega dt d\nu \tag{3}$$

The proportionality constant has to have dimension of intensity, and whatever constant present can be absorbed into σ_ν. Thus,

$$dE_\nu = I_\nu (ndA'ds\sigma_\nu) d\omega dt d\nu \tag{4}$$

Comparing both,

$$dI_\nu = -n\sigma_\nu I_\nu ds \tag{5}$$

So

$$\alpha_\nu = n\sigma_\nu = \rho\kappa_\nu \tag{6}$$

Opacity is the resistency of matter to the passage of radiation.

$$dI_\nu = -\kappa_\nu \rho I_\nu ds \rightarrow \frac{dI_\nu}{I_\nu} = -\kappa_\nu \rho ds \tag{7}$$

$$I_\nu = I_\nu(0) \exp\left\{-\int \kappa_\nu \rho ds\right\} \tag{8}$$

Define the **optical depth** infinitesimal $d\tau_\nu = \kappa_\nu \rho ds$. Optical depth is the path integral of the opacity. So, the absorption is

$$I_\nu = I_\nu(0) e^{-\tau_\nu} \tag{9}$$

Radiation is attenuated as it travels through the medium. The atenuation increases rapid;y with optical depth. For $\tau = 5$ only 0.007 of the original radiation escapes. For $\tau = 10$ is it 5×10^{-5}. In general, τ is the number of mean free paths of a photon, and we see down to only to $\tau \approx 1$.

1.2 Optical depth as mean free path

Let us show that $\tau_\nu = 1$ means one photon mean free path.

$$\tau_\nu = 1 = \int_0^{\bar{l}} \kappa_\nu \rho ds \tag{10}$$

The average optical depth over which a photon travels before being absorbed.

$$< \tau_\nu >= \int_0^\infty \tau_\nu p(\tau_\nu) d\tau_\nu \tag{11}$$

where $p(\tau_\nu)d\tau_\nu$ is the probability of being absorbed in the interval $(\tau_\nu, \tau_\nu + d\tau_\nu)$, after having traveled $(0, \tau_\nu)$, before being absorbed. In other words, it is the probability of *not* being absorbed in $(0, \tau_\nu)$ *and* being absorbed in $d\tau_\nu$.

Probability of absorption:

$$p = \frac{\Delta I(\tau_\nu)}{I_0} \tag{12}$$

the equation above is 0 for ΔI = 0 (not absorbed) and 1 for ΔI = I (absorbed).

$$p = \frac{\Delta I(\tau_\nu)}{I_0} = \frac{I_0 - I(\tau_\nu)}{I_0} = 1 - \frac{I_\nu(\tau_\nu)}{I_0} \tag{13}$$

Probability p_1 that the photon is not absorbed until τ_ν: $p_1 = 1 - p = I_\nu(\tau_\nu)/I_0 = e^{-\tau_\nu}$.
Probability p_2 that the photon is absorbed in $(\tau_\nu, \tau_\nu + d\tau_\nu)$ is

$$p_2 = \frac{\Delta I(\tau_\nu, \tau_\nu + d\tau_\nu)}{I_\nu(\tau_\nu)} \tag{14}$$

$$= \frac{dI_\nu}{I_\nu(\tau_\nu)} = d\tau_\nu \tag{15}$$

So, the total probability is $p_1 \times p_2$

$$\text{total probability} : \left(\begin{array}{c} \text{Not absorbed in } [0, \tau_\nu] \\ e^{-\tau_\nu} \end{array} \right) \times \left(\begin{array}{c} \text{absorbed in } [\tau_\nu, \tau_\nu d\tau_\nu] \\ d\tau_\nu \end{array} \right) = e^{-\tau_\nu} d\tau_\nu \tag{16}$$

Therefore,

$$< \tau_\nu > \;=\; \int_0^\infty \tau_\nu p(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu \exp(\tau_\nu) d\tau_\nu \tag{17}$$

$$= -(1 + x)e^{-x}|_0^\infty = 0 + (1)e^0 = 1 \tag{18}$$

The average optical depth traveled before being absorbed is $< \tau_\nu >$=1
For a homogeneous material, $\kappa\rho \equiv$ const

$$< \tau_\nu >= 1 = \int \kappa \rho dz = \kappa \rho l_{\text{mfp}} \tag{19}$$

So

$$l_{\text{mfp}} = \frac{1}{\kappa \rho} \tag{20}$$

1.3 Emission

Emission is parametrized as energy per unit time per unit solid angle per unit volume.

$$\begin{aligned} dE_\nu &= j_\nu \rho dV d\omega dt d\nu & (21) \\ &= j_\nu \rho dA ds d\omega dt d\nu & (22) \end{aligned}$$

so,

$$dI_\nu = j_\nu \rho ds \tag{23}$$

with $dI_\nu > 0$.

1.4 The radiative transfer equation

Combine absorption and emission

$$dI_\nu = -\kappa_\nu \rho I_\nu ds + j_\nu \rho ds \tag{24}$$

$$\frac{dI_\nu}{\kappa_\nu \rho ds} = -I_\nu + \frac{j_\nu}{\kappa_\nu} \tag{25}$$

Use the definition of optical depth, and define $S_\nu \equiv j_\nu / \kappa_\nu$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{26}$$

This is the fundamental equation of radiation transfer. The function S_ν is called the **source function**.

Notice that in the special case of blackbody radiation, $I_\nu = B_\nu$ and $dI_\nu = 0$, so $S_\nu = B_\nu$.

The general solution of the equation is found by using the integration factor e^τ_ν on both sides.

$$e^\tau_\nu dI_\nu + e^\tau_\nu I_\nu d\tau_\nu = S_\nu e^\tau_\nu d\tau_\nu \tag{27}$$

$$d(e^\tau_\nu I_\nu) = S_\nu e^\tau_\nu d\tau_\nu \tag{28}$$

And integrating between 0 and τ_ν

$$e^{\tau_\nu} I_\nu(\tau_\nu) - I_\nu(0) = \int_0^{\tau_\nu} S_\nu(t_\nu) e^{t_\nu} dt_\nu \tag{29}$$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(\tau_\nu - t_\nu)} dt_\nu \tag{30}$$

The 1st term is the extinction of the original intensity, and the 2nd is the emission in a point t_ν, extinguished in the path from t_ν to τ_ν.

1.5 Limits

Let us consider some limits that will illustrate some cases of physical interest. Consider first the *optically thin* limit, of $\tau_\nu \ll 1$. In this case we can approximate

$$e^{-\tau_\nu} \rightarrow 1 - \tau_\nu \tag{31}$$

and thus

$$I_\nu(L) = I_\nu(0) + j_\nu \rho L \tag{32}$$

The result shows that in the optically thin case, the original intensity is unaltered, but emission was added. The emission is unnatenuated, so all contributions from 0 to the full geometrical depth L are added. In terms of the optical depth, this becomes

$$I_\nu(\tau_\nu) = I_\nu(0) + S_\nu \tau_\nu \tag{33}$$

In the *optically thick* case, $\tau_\nu \gg 1$, the equation of radiative transfer becomes

$$I_\nu = S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{j_\nu}{\kappa_\nu}\frac{\kappa_\nu \rho}{\kappa_\nu \rho} = j_\nu \rho \left(\frac{1}{\kappa_\nu \rho}\right) \tag{34}$$

$$I_\nu = j_\nu \rho \bar{l} \tag{35}$$

Meaning that only stuff at 1 mean free path contributes to the emission.

2 Kirchhoff three laws of spectroscopy

Gustav Robert Kirchhoff, not knowing about energy levels in the atom, coined the term "blackbody" radiation and also postulated three empirical laws that take his name.

1. A hot dense gas produces light with a continuous spectrum.
2. A hot tenuous gas produces light with emission lines at discrete wavelengths
3. A hot dense gas surrounded by a cool tenuous gas produces light with a continuous spectrum which has gaps at discrete wavelengths.

An illustration of the laws is shown in fig 1. Let us understand them in the light of the equation of radiative transfer.

Consider the case $I_\nu(0) = 0$. No light shining, only hot gas.

$$I_\nu = S_\nu(1 - e^{-\tau_\nu}) \tag{36}$$

The 1st law is derived from this in the optically thick case. Simply putting $\tau_\nu \gg 1$. Leads to $I_\nu = S_\nu$. As we know that for a hot object or gas is a blackbody, S_ν is the Plank function, so $I_\nu = B_\nu$.

The 2nd law is also derived from this, but in the optically thin case, $e^{-\tau_\nu} = 1 - \tau_\nu$, so $I_\nu = S_\nu \tau_\nu$. The intensity will be high where τ_ν is high. Since there is no background intensity, these are seen as emission lines.

The 3rd law is the case where there is a background intensity but no emission. For the hot gas the intensity is B_ν, where the cold gas has $S_\nu = 0$. Thus $I_\nu = B_\nu e^{-\tau_\nu}$. The intensity is a Planck continuum, lowered where τ_ν is high. Thus, we see absorption lines.

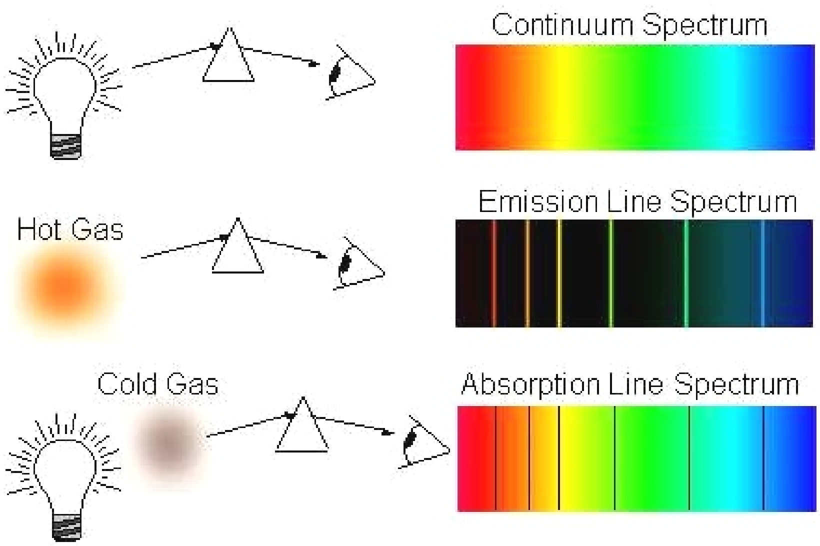

Figure 1: Illustration of the 3 empirical Kirchhoff laws of spectroscopy.