Class notes - 2 (Aug 26th, 2015)

1 Principles of Radiative Transfer

The theory of radiative transfer is a macroscopic description of radiation fields. One of the most primitive concepts is that of the energy flux, an intuitive measurement of energy flow. It is defined as the energy passing through an area over time

$$dE \propto dAdt \tag{1}$$

where the flux is the proportionality factor, so that

$$dE = FdAdt \tag{2}$$

We can also define a frequency range, so that the flux is the net energy crossing a unit area per unit time in a frequency range.

$$dE_\nu = F_\nu dAdtd\nu \tag{3}$$

Considering an isolated star, if we put spherical surfaces s and $\mathcal{S}$ of radius r and R around it, we know by conservation of energy that the total energy passing through s and $\mathcal{S}$ must be the same. Thus,

$$F(r)4\pi r^2 = F(R)4\pi R^2, \tag{4}$$

or

$$F(r) = F(R)\left(\frac{R}{r}\right)^2. \tag{5}$$

If we consider R as the radius of the star and r an arbitrary location away from the stellar surface, Eq. (**??**) says that the flux falls with the square of the distance. This is merely a statement of conservation of energy.

1.1 Intensity

The flux is a measurement of *all* rays that pass through a given area. A more fundamental description of the radiation field should consider the energy coming from individual rays. If we were dealing with quantum mechanics, that would be a single photon. However, as said, the radiative transfer theory is a classical (or semi-classical) theory, useful for dealing with radiation fields in a macroscopic way. We wish then to consider the energy coming from individual rays. However, a single ray carries essentially no energy, so we need to consider the energy carried by a *set* of rays, that differ infinitesimally from the given ray. For that, construct an area $d\boldsymbol{A}$ normal to the direction of the ray, and consider all rays that pass though $d\boldsymbol{A}$ whose direction is within a solid angle $d\boldsymbol{\omega}$ of the given ray.

$$dE \propto d\boldsymbol{A} \cdot d\boldsymbol{\omega} dt \tag{6}$$

and the proportionality is called specific intensity, or simply intensity. As with flux, we can also define it in a frequency range

$$dE_\nu = I_v d\boldsymbol{A} \cdot d\boldsymbol{\omega} dt d\nu \tag{7}$$

or

$$I_v = \lim_{\Delta \to 0} \frac{dE_\nu}{d\boldsymbol{A} \cdot d\boldsymbol{\omega} dt d\nu} \tag{8}$$

Infinitesimally, the set of rays approach, in the limit, the energy of a single ray. The intensity has dimension of

$$[I_\nu] = \text{energy (time)}^{-1} \text{ (area)}^{-1} \text{ (solidangle)}^{-1} \text{ (frequency)}^{-1} \tag{9}$$
$$= \text{ergs s}^{-1} \text{ cm}^{-2} \text{ ster}^{-1} \text{ Hz}{-1} \tag{10}$$

Notice that both the area and the solid angle infinitesimals are vectors, pointing to their respective normals, $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$. Defining

$$d\boldsymbol{A} = \hat{\boldsymbol{n}} dA \tag{11}$$
$$d\boldsymbol{\omega} = \hat{\boldsymbol{n}}' d\omega \tag{12}$$
$$\tag{13}$$

we have

$$dE_\nu = I_v \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}' dA d\omega dt d\nu = I_v \cos\theta dA d\omega dt d\nu \tag{14}$$

where theta is the plane angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

1.1.1 Constancy of intensity along a ray

When we take the limit $\Delta\omega \to 0$, the beam stops spreading out, approaching a single ray. Due to energy conservation, the energy is constant along a ray, and thus the intensity does not depend on distance. To prove this, consider a ray L, and choose two points along it, P1 and P2, separated by a distance r. Then construct area elements around P1 and P2, call them dA_1 and dA_2, and consider the solid angle $\mathrm{d}\omega_1$ that goes from $\mathrm{d}A_1$ to $\mathrm{d}A_2$. The energy flowing through this area into this solid angle is

$$dE_1 = I_1 dA_1 d\omega_1 dt \tag{15}$$

where $d\omega_1$ is the angular size of dA_2 as seen from dA_1, i.e., $d\omega_1 = dA_2/r^2$.
The energy that crosses $\mathrm{d}A_2$ coming from $\mathrm{d}A_1$ is conversely

$$dE_2 = I_2 dA_2 d\omega_2 dt \tag{16}$$

where $d\omega_2 = dA_1/r^2$ is the angular size of dA_1 as seen from dA_2. Because energy is conserved, $dE_1 = dE_2$, and we have

$$I_1 dA_1 d\omega_1 = I_2 dA_2 d\omega_2 \tag{17}$$

Substituting the definitions of the solid angles, we conclude that $I_1 = I_2$, and the intensities at P1 and P2 are equal.
This means that the surface of the Sun is as bright seen from Earth as it is seen from Neptune. This may seem counterintuitive at first, until we realize that a wall does not get brighter as we approach it. Intensity is an intrinsic property of *resolved* sources of radiation.

1.2 Flux

Flux is the net energy in a frequency interval that passes per unit time through a unit area in a frequency range

$$F_\nu = \frac{dE_\nu}{dA d\nu dt}. \tag{18}$$

Comparing that with the definition of intensity, we have that

$$F_\nu = \int I_\nu \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}' d\omega = \int I_\nu cos\theta d\omega \tag{19}$$

i.e., the flux is the directional integral of the intensity. For an isotropic field, where I_ν does not depend on direction

$$F_\nu = I_\nu \oint \cos\theta d\omega = I_\nu \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta = \pi I_\nu \tag{20}$$

The flux is integrated in solid angle, so the directional information is eliminated. The intensity can only be defined for extended objects, angularly resolved, because we need to specify the interval of solid angles. Flux instead can be defined for any source, including point sources. Therefore, there is a transition between measuring intensity and flux as we distance an observer from a source.

1.2.1 Bolometric Flux, Stefan-Boltzmann law, and stellar luminosity

The bolometric flux is the flux integrated in all wavelengths.

$$F_{\text{bol}} = F = \int_0^\infty F_\nu d\nu \tag{21}$$

$$F = \frac{dE}{dAdt} \tag{22}$$

The quantity dE/dt in physics is called power, yet in connection to stars we prefer to call it *luminosity*. So, Flux $\times$ Area = Luminosity.
For a source of constant luminosity, as most sources in astrophysics are (in the timescales we measure them), the product Flux $\times$ Area is constant. Therefore, the flux falls with area following an inverse square law, recovering Eq. (**??**).
Notice that the luminosity can be determined knowing the radius of the star and the flux in its surface.

$$L_\star = \text{Area} \times \text{Flux} = 4\pi R^2 \times F_\star \tag{23}$$

where we have to determine the flux at the surface of the star. Approximating a star as a blackbody, we can write

$$I_\nu = B_\nu \tag{24}$$

where B_ν is Planck's law

$$B_\nu = \frac{2\nu^2}{c^2} \frac{h\nu}{\exp\left(h\nu/KT\right) - 1} \tag{25}$$

Because we only see one hemisphere of the star, we need to integrate through the half-sphere.

$$F_\nu = \int_0^{2\pi} \int_0^{\pi/2} B_\nu \cos\theta \sin\theta d\theta d\phi = \pi B_\nu \tag{26}$$

For the bolometric flux we integrate in frequency

$$F = \int_0^\infty B_\nu d\nu = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3 d\nu}{\exp\left(h\nu/KT\right) - 1} \tag{27}$$

Substituting $x = h\nu/KT$, we have

$$F = \frac{2\pi h}{c^2} \left(\frac{KT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \tag{28}$$

The integral can be evaluated to $\pi^4/15$. The result is thus

$$F = \left(\frac{K^4 \pi^2}{60 c^2 \hbar^3}\right) T^4 = \sigma T^4 \tag{29}$$

This result is called *Stefan-Boltzmann law*. The constant that appears

$$\sigma = \left(\frac{K^4 \pi^2}{60 c^2 \hbar^3}\right) \tag{30}$$

is the Stefan-Boltzmann constant. Its numerical value is 5.6704×10^{-5} erg cm^{-2} s^{-1} K^{-4}. Based on this, we can write the stellar luminosity

$$L_\star = 4\pi R_\star^2 \sigma T_\star^4. \tag{31}$$

1.3 Wien's displacement law

Wien's displacement law states that the peak wavelength of a blackbody radiation, multiplied by its temperature, is constant.

$$\lambda_{\max} T \equiv \text{const} = 2.898 \times 10^{-3} mK \tag{32}$$

This law follows immediately from Planck's curve, by taking the derivative wrt frequency (or wavelength)

$$\frac{c^2}{2h} \frac{dI_\nu}{d\nu} = \frac{3\nu^2}{\exp\left(h\nu/KT\right) - 1} - \frac{\nu^3}{\left(\exp\left(h\nu/KT\right) - 1\right)^2} e^{h\nu/KT} \frac{h}{KT} \tag{33}$$

and equating it to zero, we find the frequency ν_M where the intensity peaks.

$$\frac{\nu_M^2}{\exp\left(h\nu_M/KT\right) - 1} \left(3 - \frac{\nu_M \exp\left(h\nu_M/KT\right)}{\left(\exp\left(h\nu_M/KT\right) - 1\right)} \frac{h}{KT}\right) = 0 \tag{34}$$

Substituting $x = h\nu_M/KT$, it reduces to

$$xe^x - 3e^x + 3 = 0 \tag{35}$$

which we can solve numerically to find $x \equiv$ const, i.e, $\nu_M/T \equiv$ const and thus $\lambda_M T \equiv$ const as in Wien's law.
Eq. (**??**) and Eq. (**??**) already explain a lot of features of the HR diagram. The luminosity is a function of the radius (more emitting surface), so dwarfs are less luminous than giants. Betelgeuse is about 1000 times bigger than the Sun, so the area by itself would account for a million-fold factor in luminosity. The luminosity is also a very strong function of the temperature. A change in temperature by a factor 10, as in the main sequence from M to O stars, implies a 10^4 change in luminosity.

1.4 Magnitudes

The flux depends on distance, so the magnitudes we measure are apparent. For absolute magnitude, we need a standard distance, that we define as D=10 pc.

$$m = -2.5 \log F + C \tag{36}$$

with

$$F = \frac{L_\star}{4\pi d^2} \tag{37}$$

The absolute magnitude is

$$M = -2.5 \log \left(\frac{L_\star}{4\pi d^2} \right) + C \tag{38}$$

We can define also the distance modulus, which is the difference between apparent and absolute magnitude

$$m - M = -2.5 \log \left(\frac{L_\star}{4\pi d^2} \frac{4\pi D^2}{L_\star} \right) = 5 \log \left(\frac{d}{10\text{pc}} \right) \tag{39}$$

The distance is thus

$$d(\text{pc}) = 10^{0.2(m-M)+1} \tag{40}$$

It is also useful to express this as

$$M = m + 5 \log \pi^{''} + 5 \tag{41}$$

where $\pi^{''}$=1/d(pc) is the parallax angle (do not confuse it with the usual circle-related number!). The equation above depends only on measurable quantities and is useful to have in handy when observing.

1.5 Mean intensity

We can define also the mean intensity J_ν, which is the directional average of the intensity

$$J_\nu = \frac{\oint I_\nu d\omega}{\oint d\omega} \tag{42}$$

$$= \frac{1}{4\pi}\oint I_\nu d\omega \tag{43}$$

This quantity is related to the energy density of the radiation field. Consider it the energy per unit volume in a frequency range. It is useful as a step to define it also per unit solid angle

$$dE_\nu = u_{\nu,\omega} dV d\nu d\omega \tag{44}$$

The volume defined by a light beam is $dA\, c\, dt$, where $c\, dt$ is the distance light travels in a unit time dt. So,

$$dE_\nu = u_{\nu,\omega}\, c\, dA\, dt\, d\nu\, d\omega = I_\nu dA\, dt\, d\nu\, d\omega \tag{45}$$

Comparing both identities, we find $u_{\nu,\omega} = I_\nu / c$, and integrating it in solid angle

$$u_\nu = \frac{1}{c}\oint I_\nu d\omega \tag{46}$$

$$u_\nu = \frac{4\pi}{c} J_\nu \tag{47}$$

For a black body,

$$u = \frac{4\pi}{c}\int B_\nu d\nu = \frac{4\sigma}{c} T^4 = aT^4 \tag{48}$$

The constant

$$a = \frac{4\sigma}{c} = 7.5667 \times 10^{-15} \text{ergs cm}^{-3}\ \text{K}^{-4} \tag{49}$$

is called the *radiation constant*.

1.6 Solved problem

A spherical star of radius R emits radiation of intensity I_ν, in all directions. For a distance r, describe the radiation field, obtain the mean intensity, and the flux.

The observer sees an anisotropic radiation field, with the star subtending a finite angular size, and elsewhere having zero intensity. The field is thus

$$I_\nu(\theta,\phi) = \begin{cases} I_\nu & \text{for} \quad 0 \le \theta \le \theta_M, \quad 0 \le \phi \le 2\pi \\ 0 & \text{for} \quad \theta_M < \theta \le \pi/2, \quad 0 \le \phi \le 2\pi \end{cases} \tag{50}$$

where $\theta_M = \text{asin}(R/r)$.

The mean intensity is

$$J_\nu = \frac{1}{4\pi}\oint I_\nu d\omega = \frac{I_\nu}{4\pi}\int_0^{2\pi} d\phi \int_0^{\theta_M} \sin\theta d\theta \tag{51}$$

$$= \frac{1}{2} I_\nu (1-\cos\theta_M) = \frac{1}{2} I_\nu \left(1 - \cos\left(\mathrm{asin}(R/r)\right)\right) \tag{52}$$

using $(1-\cos(\mathrm{asin}(R/r))) = \sqrt{1-\sin^2(\mathrm{asin}(R/r))} = \sqrt{1-(R/r)^2} = 1/r(r^2-R^2)^{-1/2}$

$$J_\nu = \frac{I_\nu}{2r}(r-(r^2-R^2)^{-1/2}). \tag{53}$$

The flux is

$$F_\nu = \int I_\nu cos\theta d\omega = I_\nu \int_0^{2\pi} d\phi \int_0^{\theta_M} \cos\theta \sin\theta d\theta \tag{54}$$

$$= \pi I_\nu \sin^2\theta_M = \pi I_\nu \sin^2(\mathrm{asin}(R/r)) \tag{55}$$

$$= \pi I_\nu \left(\frac{R}{r}\right)^2 \tag{56}$$