Cosmology notes: Class 2 (Dec 2nd, 2015)

Last class we saw that as $z \to \infty$, $\Omega \to 1$, so the early Universe was essentially flat. This allows us to study the early Universe using simplified yet accurate equations by setting $k = 0$ in the Friedmann equation.

For $k = 0$, the Friedmann equation becomes

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G\rho_{c,0}}{3R} \tag{1}$$

This can be easily integrated

$$\int_0^R \sqrt{(R')}dR' = \sqrt{8\pi G\rho_{c,0}}3\int_0^t dt' \tag{2}$$

which yields

$$R_{\rm flat} = \left(6\pi G\rho_{c,0}\right)^{1/3} t^{2/3}. \tag{3}$$

This can be simplified by substituting

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \tag{4}$$

and also by defining the *Hubble time*, which is the inverse of the Hubble constant

$$t_H = H_0^{-1} \tag{5}$$

With these,

$$R_\flat = \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3} \tag{6}$$

This yields the scale factor of a flat universe filled with dust at all times.

Closed and open Universe

If $\Omega \neq 1$ the Friedmann equation is solvable, though the solution is more involved. The algebra is tedious, so we simply quote the result.

For a closed universe,

$$R_{\rm closed} = \frac{4\pi G\rho_0}{3kc^2}\left[1 - \cos(x)\right] \tag{7}$$

where $x = t/t_H$. The closed universe has infinite bounces, which are mathematical artifacts.

For an open universe,

$$R_{\rm open} = \frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)}\left[\cosh(x) - 1\right]. \tag{8}$$

Age of the Universe

These results for $R(t)$ can be inverted to yield the age of these model Universes. For the flat universe

$$R_{\text{flat}} = \left(\frac{3}{2}\right)^{2/3} \left(\frac{t}{t_H}\right)^{2/3} \tag{9}$$

Substitute $R = 1/(1+z)$ to yield

$$\frac{t(z)}{t_H} = \frac{2}{3} \frac{1}{(1+z)^{3/2}} \tag{10}$$

For the current age of the Universe, set $z = 0$. So,

$$t_0 = \frac{2}{3} t_H \approx 9.2 \text{ Gyr} \tag{11}$$

The age is too low because this is the age of a Universe filled with pressureless dust. But there is more to the Universe than pressureless dust. Photons, for instance, outnumber baryons by two billion to one. Let us include the effect of radiation in the Friedmann equation.

Pressure

We have considered only that the shell has kinetic and potential energy, for the case of pressureless dust. Assuming that this density will produce pressure, let us apply the first law of thermodynamics

$$dU = dQ - dW \tag{12}$$

The Universe has the same temperature all over, so there is no heat flux, $dQ = 0$ (the expansion is adiabatic). So

$$dU = -dW = -PdV \tag{13}$$

And dividing by dt,

$$\frac{dU}{dt} == -P\frac{dV}{dt} \tag{14}$$

Since $V = 4\pi r^3/3$,

$$\frac{dU}{dt} == -\frac{4\pi}{3} P \frac{dr^3}{dt} \tag{15}$$

To remove the factor we can define the internal energy per unit volume, $u = U/(4/3\pi r^3)$, so

$$\frac{d\left(r^3 u\right)}{dt} == -P\frac{dr^3}{dt} \tag{16}$$

Writing $u \equiv \rho c^2$ yields

$$\frac{d\left(r^3 \rho\right)}{dt} == -\frac{P}{c^2}\frac{dr^3}{dt} \tag{17}$$

Now using $r = R\omega$, we obtain the fluid equation

$$\frac{d\left(R^3\rho\right)}{dt} == -\frac{P}{c^2}\frac{dR^3}{dt} \tag{18}$$

This equation can be used to determine the pressure of different components of the Universe or, conversely, how ρ changes with time. For pressureless dust, $d/dt(R^3\rho) = 0$, so we get $R^3\rho = \text{const.}$ Making use now of the Friedmann equation

$$\left[\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\rho\right] R^2 = -kc^2 \tag{19}$$

Multiply this by R and take the time-derivative. Then use Eq. (18) to substitute $d/dr(R^3\rho)$, and Eq. (35) to substitute kc^2. With these, we arrive at the acceleration equation

$$\frac{d^2R}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) R \tag{20}$$

We see from this equation that the pressure slows down the expansion (for $P > 0$). This is counter-intutitive, since we are used to pressure making thing expand. In fact, this notion does not apply in cosmology. In usual fluid mechanics, it is the pressure gradient that enters in the momentum equation. In cosmology there is no pressure gradient to exert force because inner and outer pressures are the same in the shell. What is happening is that, due to energy-mass equivalence, the gravity of the fluid's kinetic energy is providing the slow-down.

To solve the fluid and acceleration equations, we need an equation of state. We can write

$$P = wu = w\rho c^2 \tag{21}$$

where w is a factor to be determined for each component. We already saw that for dust (matter and dark matter), $w_m = 0$. For radiation, $P_{\text{rad}} = u_{\text{rad}}/3$, so $w_{\text{rad}} = 1/3$. If we plug that into the fluid equation

$$R^{3(1+w)}\rho \equiv \text{const} = \rho_0 \tag{22}$$

According to this equation with $w_{\text{rad}} = 1/3$ and $R(t_0) = 1$

$$R^4 u_{\text{rad}} = u_{\text{rad},0} \tag{23}$$

A factor R^3 is from volume, and another R from long-wavelength photons being redshifted, thus having less energy. Substituting $u = aT^4$

$$R^4 a T^4 = a T_0^4 \tag{24}$$

i.e.,

$$RT = T_0 \tag{25}$$

where T_0 is the current temperature of the Universe. This equation shows that the temperature of the Universe decreases linearly with the expansion. In terms of redshift,

$$T = T_0(1+z) \tag{26}$$

The CMBR is at $z \approx 1000$ and $T \approx 3000\,K$, the temperature equivalent to $T = 13.6eV$. We can calculate from this the current temperature of the Universe as $T_0 = 3K$. The CMBR was released when the Universe was a thousandth of its present size.

Radiation-dominated era

In relativity there is a mass-energy equivalence; in a model with only dust, dust's gravity was slowing down the expansion. Yet, CMB photons also gravitate, and that gravity dominated the early Universe. Considering

$$P = \omega u \tag{27}$$

the different components of the Universe dilute differently with the expansion, according to

$$R^{3(1+\omega)}\rho \equiv \text{const} \tag{28}$$

With $w = 0$ for matter and $w = 1/3$ for radiation, these dilute as

$$R^4\rho_{\text{rad}} = \rho_{\text{rad,0}} \tag{29}$$
$$R^3\rho_{\text{m}} = \rho_{\text{m,0}} \tag{30}$$
$$\tag{31}$$

And ρ_{rad} increases more rapidly with R than ρ_m as R decreases. As some point ρ_{rad} dominates, which was the radiation-dominated era. The transition from radiation era to matter era occurred when the scale factor satisfied $\rho_{\text{rad}} = \rho_m$, or $\Omega_{\text{rad}} = \Omega_m$. This happened when

$$R_{r,m} = \frac{\Omega_{\text{rad},0}}{\Omega_{m,0}} = 3.05 \times 10^{-4} \tag{32}$$

$$z_{r,m} = \frac{1}{R_{r,m}} - 1 = 3270 \tag{33}$$

$$T_{r,m} = 8920\,K \tag{34}$$

That is, before recombination. Let us now calculate the age of the Universe considering the presence of radiation. Remember that a dust-only Universe gave an age of 9 Gyr.

We write the Friedmann equation with dust and radiation

$$\left[\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3R}\left(\rho_{m,0} + \rho_{\text{rad},0}\right)\right] = 0 \tag{35}$$

We set $k = 0$ in the RHS because the early Universe was flat. Integrating the equation

$$\int_0^R \frac{R' dR'}{\left(\rho_{m,0} + \rho_{\text{rad},0}\right)^{1/2}} = \frac{8\pi G}{3}^{1/2} \int_0^t dt' \tag{36}$$

which yields

$$t(R) = \frac{2}{3}\frac{R_{r,m}^{3/2}}{H_0\sqrt{\Omega_{m,0}}}\left[2 + \left(\frac{R}{R_{r,m} - 2}\right)\sqrt{\frac{R}{R_{r,m}} + 1}\right] \tag{37}$$

Now set $R = R_{r,m}$ to find the transition time

$$t_{r,m} = 1.74 \times 10^{12}\ s = 5 \times 10^4\ \text{yr}$$

The limit $R \gg R_{r,m}$ corresponds to matter-dominated, for which

$$t(R) = \frac{2}{3} \frac{R^{3/2}}{H_0 \sqrt{\Omega_{m,0}}}, \tag{38}$$

yielding $R(t) \propto t^{2/3}$.

If $R \ll R_{r,m}$ we are at the radiation-dominated era, for which $R(t) \propto t^{1/2}$, i.e., the expansion was slower. Recasting this in terms of redshift,

$$\frac{t(z)}{t_H} = \frac{2}{3} \frac{1}{(1+z)^{3/2} \sqrt{\Omega_{m,0}}} \tag{39}$$

At $z = 0$, this yields an age of 12.5 Gyr.

Dark Energy

The Friedmann equation with a cosmological constant is

$$\left[\left(\frac{\dot{R}}{R} \right)^2 - \frac{8\pi G}{3} \rho - \frac{1}{3} \Lambda c^2 \right] R^2 = -kc^2 \tag{40}$$

The extra term can be thought of as potential energy, with

$$U_\Lambda = -\frac{1}{6} \Lambda m c^2 r^2 \tag{41}$$

and associated force

$$F_\Lambda = \frac{1}{3} \Lambda m c^2 r \hat{r} \tag{42}$$

So that the mechanical energy of a shell is

$$\frac{1}{2} m v^2 - \frac{G M_r m}{r} - \frac{1}{6} \Lambda m c^2 r^2 = -\frac{1}{2} m k c^2 \omega^2 \tag{43}$$

Notice that the force is directed outward, i.e., it is a centrifugal force, exerting a repulsive force on the shell and accelerating its expansion.

The full Friedmann equation with matter (dust), radiation, and dark energy (cosmological constant) is

$$\left[\left(\frac{\dot{R}}{R} \right)^2 - \frac{8\pi G}{3} (\rho_m + \rho_{\text{rad}}) - \frac{1}{3} \Lambda c^2 \right] R^2 = -kc^2 \tag{44}$$

Combining it with the fluid equation

$$\frac{d(R^3 \rho)}{dt} = -\frac{P}{c^2} \frac{dR^3}{dt} \tag{45}$$

yeilds

$$\frac{dR^2}{dt^2} = \left\{ -\frac{4\pi G}{3} \left[\rho_m + \rho_{\text{rad}} + \frac{3(P_m + P_{\text{rad}})}{c^2} \right] + \frac{1}{3} \Lambda c^2 \right\} R \tag{46}$$

Due to the equivalence of mass and energy we can write a density of dark energy

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} \tag{47}$$

And we can write more compactly

$$\left[\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda)\right] R^2 = -kc^2 \tag{48}$$

Notice that all the factors in the dark energy density are constant. Nothing in it changes in time as the universe expands. So,

$$\rho_\Lambda = \text{const} = \rho_{\Lambda,0} \tag{49}$$

As the other densities dilute with the expansion, dark energy eventually dominates.

Dark energy pressure

From the fluid equation we can derive the pressure due to dark energy

$$\frac{d(R^3\rho_\Lambda)}{dt} = -\frac{P_\Lambda}{c^2}\frac{dR^3}{dt} \tag{50}$$

setting $\rho_\Lambda = \text{const}$ and bringing it outside the time derivative

$$P_\Lambda = -\rho_\Lambda c^2 \tag{51}$$

i.e., $\omega_\Lambda = -1$. Dark energy exerts negative pressure.
The acceleration equation is thus

$$\frac{dR^2}{dt^2} = \left\{-\frac{4\pi G}{3}\left[\rho_m + \rho_{\rm rad} + \rho_\Lambda + \frac{3}{c^2}(P_m + P_{\rm rad} + P_\Lambda)\right]\right\} R \tag{52}$$

with the Friedmann equation in compact form

$$H^2\left[1 - (\Omega_m + \Omega_{\rm rad} + \Omega_\Lambda)\right] R^2 = -kc^2 \tag{53}$$

with

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2} \tag{54}$$

If we define $\Omega = \Omega_m + \Omega_{\rm rad} + \Omega_\Lambda$ then

$$H^2\,(1-\Omega)\,R^2 = -kc^2 \tag{55}$$

Measurements observe

$$\Omega_m = 0.27 \tag{56}$$

$$\Omega_{\rm rad} = 8.24 \times 10^{-5} \tag{57}$$

$$\Omega_\Lambda = 0.73 \tag{58}$$

These numbers are understood in light of the different dilutions laws

$$\rho_m \propto R^{-3} \tag{59}$$
$$\rho_{\text{rad}} \propto R^{-4} \tag{60}$$
$$\rho_\Lambda \propto R^0 \quad (\text{const}) \tag{61}$$

The Λ-era started at the transition when $\rho_m = \rho_\Lambda$. This occured when

$$R = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = 0.72 \tag{62}$$

This corresponds to $z = 0.39$, or 5 billion years ago.