Cosmology notes: Class 1 (Nov 30th, 2015)

Cosmology is the study of the Universe as a whole. Since the Universe is everything that exists, no other discipline can claim a more encompassing subject of study.

As gravity is the only force that survives in the large scales of cosmology (even magnetic fields get feeble in comparison, since the magnetic dipole field scales at $1/r^3$), the tools of cosmology are those of general relativity. However, significant insight can be gained from a Newtonian approach, which is the one we will take. This "Newtonian cosmology" does not have too misguided a name. In fact, had Isaac Newton known about the expansion of the Universe, he could have derived the main results of modern cosmology. Indeed, as we will promptly see, these results arise a) the cosmological principle and b) conservation of energy combined with Hubble's law. This law which states that a galaxy at a distance r from Earth recedes from us with velocity

$$v = Hr, \tag{1}$$

where H is called Hubble constant.

The cosmological principle states that the Universe is homogeneous and isotropic. If isotropy holds, all galaxies see the same Hubble law, as if they were the center. This is a straightfoward result from simple vector algebra. Consider two galaxies A and B away from Earth from a distance r_A and r_B, respectively. Hubble's law states that the recessional velocity of these galaxies as measured from Earth is

$$\boldsymbol{v}_A = H_0 \boldsymbol{r}_A \tag{2}$$

$$\boldsymbol{v}_B = H_0 \boldsymbol{r}_B \tag{3}$$

Substracting one from the other we have the recessional velocity of galaxy B as seen from galaxy A

$$\boldsymbol{v}_B - \boldsymbol{v}_A = H_0(\boldsymbol{r}_B - \boldsymbol{r}_A) \tag{4}$$

So, the observer in galaxy A sees other galaxies in the Universe moving away with the same Hubble law as Earth.

Homologous expansion

Notice that the cosmological principle also implies that, if we consider the Universe as composed of concentric shells, each of radius r, the expansion is the same for all shells. That is to say, the expansion is homologous: all shells take the same time to double their radius. Therefore, we need only concentrate on one single shell to understand the behavior of the Universe.

Consider thus a shell of mass m at time t, expanding with the Universe with recessional velocity $v = dr/dt$. As the Universe expands, the density and radius of the shell change in time, i.e., $\rho = \rho(t)$ and $r = r(t)$. We can write the mechanical energy of the shell as

$$K(t) + U(t) = E \tag{5}$$

Given that the potential energy is the gravitational pull of the mass inside the shell, the mechanical energy is

$$\frac{1}{2}mv^2(t) - \frac{GM_r m}{r(t)} = E \tag{6}$$

According to the cosmological principle, the mass M_r inside the shell has the same density as the shell, so

$$M_r = \frac{4\pi}{3}r^3(t)\rho(t), \tag{7}$$

and we can write the mechanical energy as

$$v^2(t) - \frac{8\pi}{3}G\rho(t)r^2(t) = \frac{2E}{m}. \tag{8}$$

Scale factor

Since the radius $r(t)$ of each shell changes with time, it is convenient to define a reference radius against which to measure the expansion. I.e., we can write

$$r(t) = R(t)\omega \tag{9}$$

where $r(t)$ is the radius of the shell, called *coordinate distance*. The quantity ω does not change for a particular shell: it effectively "labels" a shell and follows its expansion. It is called *comoving coordinate*. $R(t)$ is dimensionless and is called the *scale factor*, i.e., the factor by which we have to scale the comoving coordinate to get the coordinate distance. By convention, at present time the scale factor is unity, $R(t_0) = 1$, corresponding to $r(t_0) = \omega$.

The evolution of the shell (and by consequence, the Universe), is given by the time behavior of $R(t)$. We can substitute the velocity in the mechanical energy by Hr, as given by Hubble's law

$$v(t) = H(t)r(t) = H(t)R(t)\omega \tag{10}$$

Or, alternatively,

$$v(t) = \frac{dr(t)}{dt} = \omega\frac{dR(t)}{dt} \tag{11}$$

So the Hubble constant can also be written as

$$H(t) = \frac{1}{R(t)}\frac{dR(t)}{dt} \tag{12}$$

Putting this back in the energy equation, and omitting the "(t)" for clarity, we have

$$v^2 - \frac{8\pi}{3}G\rho r^2 = \frac{2E}{m} \tag{13}$$

$$\left(H^2 - \frac{8\pi}{3}G\rho\right)R^2 = \frac{2E}{m\omega^2} \tag{14}$$

Here we can redefine the total energy to get rid of the mass of the shell m, the co-moving distance ω, as well as the factor 2. A suitable choice is

$$E = -\frac{1}{2} m c^2 k \omega^2 \tag{15}$$

where k is a constant. So,

$$\left(H^2 - \frac{8\pi}{3} G\rho \right) R^2 = -kc^2 \tag{16}$$

and we arrive then at the Friedmann equation

$$\left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8\pi}{3} G\rho \right] R^2 = -kc^2. \tag{17}$$

We can also write the Friedmann equation in terms of the present density ρ_0 (which we can measure) by making use of mass conservation

$$R(t)^3 \rho(t) = R^3(t_0)\rho(t_0) = \rho_0 \tag{18}$$

And substituting the above equality in the Friedmann equation

$$\left(\frac{dR}{dt} \right)^2 - \frac{8\pi G \rho_0}{3R} = -kc^2 \tag{19}$$

Closed, Open, or Flat

Based on the sign of the energy, the Universe has three behaviors:

Closed (bounded) Universe:	$k > 0$ (negative energy)
Open (unbounded) Universe:	$k < 0$ (positive energy)
Flat Universe:	$k = 0$ (zero energy).

For critical density, $k = 0$

$$\left(H^2 - \frac{8\pi}{3} G\rho \right) R^2 = 0 \tag{20}$$

where the equality holds when the density has the critical value

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}. \tag{21}$$

To find the numerical value of this critical density, we need to know the Hubble constant. The Hubble constant is usually written as

$$H_0 = 100\, h \,\mathrm{km\,s^{-1}\,Mpc^{-1}} = 3.24 \times 10^{-18}\, h\,\mathrm{s^{-1}} \tag{22}$$

The quantity h is historical. In the early days they could not measure H_0 precisely, but they knew the value was around 100 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$. So, they wrote it as that number, times a dimensionless factor, that embodied how far from this round number the actual value was. The original estimate was between 0.5 and 1. WMAP [1] measured $h = 0.71^{+0.04}_{-0.03}$. So

[1] The value was updated by Planck to $h = 0.678 \pm 0.0077$.

$$[H_0]_{\text{WMAP}} = 71 \text{ km s}^{-1} \text{Mpc}^{-1} = 2.30 \times 10^{-18} \text{s}^{-1} \tag{23}$$

And the present value of the critical density is

$$\rho_{c,0} = 9.47 \times 10^{-27} \text{ kg m}^{-3} \tag{24}$$

Which is roughly six hydrogen atoms per cubic meter. WMAP measured the density of visible matter in the Universe as $\rho_{b,0} = 4.17 \times 10^{-28}$ kg m^{-3}, or 4% of the critical density.

Redshift

A quantity of great interest in cosmology is the redshift, a readily observable quantity, defined as the shift in a wavelength with respect to the original wavelength

$$z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}}. \tag{25}$$

Considering that the cosmological redshift is due to Hubble's law, we can write

$$z = \frac{r_{\text{obs}} - r_{\text{emitted}}}{r_{\text{emitted}}} = \frac{r_{\text{obs}}}{r_{\text{emitted}}} - 1. \tag{26}$$

And given $r_{\text{obs}} = r_0 = \omega$, and $r_{\text{emitted}} = r(t)$, the first term in the RHS is the inverse of the scale factor

$$R = \frac{r(t)}{r_0}. \tag{27}$$

So, we can write

$$z = \frac{1}{R} - 1 \tag{28}$$

and convsersely

$$R = \frac{1}{1+z}. \tag{29}$$

Density Parameter

Consider the ratio of measured density to critical density

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G \rho(t)}{3H^2(t)} \tag{30}$$

Presently

$$\Omega_0 = \frac{\rho_0}{\rho_{c,0}} = \frac{8\pi G \rho_0}{3H_0^2} \tag{31}$$

According to WMAP, the density of matter (dark + luminous) is $\Omega_{m,0} = 0.27 \pm 0.04$. And the density of luminous matter alone is $\Omega_{b,0} = 0.044 \pm 0.004$.

We can write the ratio of the density parameter to the current density parameter as

$$\frac{\Omega}{\Omega_0} = \frac{\rho}{\rho_0}\frac{H_0^2}{H^2} \tag{32}$$

And considering the conservation of mass, $\rho/\rho_0 = 1/R^3$. Writing this in terms of the redshift,

$$\frac{\Omega}{\Omega_0} = (1+z)^3\frac{H_0^2}{H^2} \tag{33}$$

or

$$\Omega H^2 = (1+z)^3\Omega_0 H_0^2 \tag{34}$$

Substituting this in the Friedmann equation, we arrive at

$$H^2(1-\Omega)R^2 = -kc^2 \tag{35}$$

which, for present time becomes

$$H_0^2(1-\Omega_0) = -kc^2 \tag{36}$$

So, if

$\Omega_0 > 1$ then $k > 0$, $E < 0$: Universe is closed;
$\Omega_0 < 1$ then $k < 0$, $E > 0$: Universe is open;
$\Omega_0 = 1$ then $k = 0$, $E = 0$: Universe is flat.

An interesting result arises from these equations as well. Equating Eq. (35) and Eq. (36), and substituting R for z

$$H^2(1-\Omega) = H_0^2(1-\Omega_0)(1+z)^2 \tag{37}$$

Or,

$$H = H_0(1+z)\left(\frac{1-\Omega_0}{1-\Omega}\right)^{1/2} \tag{38}$$

That is, we can relate the density parameter with the present one by

$$\Omega = \left(\frac{1+z}{1+\Omega_0 z}\right)\Omega_0 = 1 + \frac{(\Omega_0 - 1)}{(1+\Omega_0 z)} \tag{39}$$

A couple of results can be derived from this equation.
First, as $z \to \infty$, $H \to \infty$. That is, as the scale factor goes to zero, the rate of expansion goes to infinity. This means that if the universe had zero size, there was a big bang.
Second, the sign of $\Omega - 1$ does not change. The Universe is either always closed, always open, or always flat. The last is particularly interesting: if $\Omega = 1$ any time, then $\Omega = 1$ at all times.
Third, as $z \to \infty$, $\Omega \to 1$. The early Universe was essentially flat.
The last one allows us to simplify several equation in the early Universe by setting $k = 0$.