#3

Magneto Hydro Dynamics (MHD)

Selected Topics in Astrophysics

Parker wind

Impossibility of hydrostatic equilibrium in the corona.
Consider the equations for momentum and temperature

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla) v - \frac{1}{\rho} \nabla p + F$$

$$\frac{\partial T}{\partial t} = -(v \cdot \nabla) T + \nabla (K \cdot \nabla T)$$

With ideal gas equation of state. If hydrostatic equilibrium applies we can use it to describe the corona

$$\frac{dp}{dr} = -\frac{GM_\odot}{r^2} \rho$$

And substitute $p = \rho \frac{K_B}{m} T$ as per the equation of state

$$\frac{dp}{dr} = -\frac{GM_\odot}{r^2} \frac{m}{K_B} \frac{p}{T}$$

$$\frac{dp}{p} = -\frac{GM_\odot}{r^2} \frac{m}{K_B} \frac{dr}{T}$$

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{GM_\odot m}{K_B T} \int_{r_0}^{R} \frac{dr}{r^2}$$

$$= \frac{GM_\odot m}{K_B T} \frac{1}{r} \Big|_{r_0}^{R} = \frac{GM_\odot m}{K_B T} \left(\frac{1}{R} - \frac{1}{r_0} \right)$$

So the pressure is

$$p = p_0 \exp\left(\frac{GM_\odot m}{K_B T}\left(\frac{1}{R} - \frac{1}{r_0}\right)\right)$$

$$p = p_0 \exp\left(\frac{GM_\odot m}{K_B T r_0}\left(\frac{r_0}{R} - 1\right)\right)$$

The above equation implies that the pressure has a non-zero assymptote as the distance R goes to infinity. Also, this assymptote is much larger than the pressure of the interstellar medium. The conclusion is we cannot get a solution where both p and T go to zero at infinity. So, the corona cannot be in hydrostatic equilibrium and must be flowing away as a wind.

Wind model

We abandon the idea of equilibrium $(v = \partial_t = 0)$ for steady state $(\partial_t = 0,\ v \neq 0)$

Applying that to the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\nabla(\rho \cdot u)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho u\right) = 0$$

$$2rv\rho + r^2\rho\frac{dv}{dr} + vr^2\frac{d\rho}{dr} = 0$$

The Euler equation becomes

$$\rho v \frac{dv}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2}\rho \qquad ; \quad p = \rho c_s^2$$

$$\frac{d\rho}{dr} = -\frac{\rho v}{c_s^2}\frac{dv}{dr} - \frac{GM}{r^2}\frac{\rho}{c_s^2}$$

$$2rv\rho + r^2\rho\frac{dv}{dr} + vr^2\frac{d\rho}{dr} = 0$$

$$\frac{d\rho}{dr} = -\frac{\rho v}{c_s^2}\frac{dv}{dr} - \frac{GM}{r^2}\frac{\rho}{c_s^2}$$

$$2rv\cancel{\rho} + r^2\cancel{\rho}\frac{dv}{dr} + vr^2\left[-\frac{\cancel{\rho} v}{c_s^2}\frac{dv}{dr} - \frac{GM}{r^2}\frac{\cancel{\rho}}{c_s^2}\right] = 0$$

$$2rv + r^2\frac{dv}{dr}\left[1 - \frac{v^2}{c_s^2}\right] - v\frac{GM}{c_s^2} = 0$$

$$\frac{dv}{dr}\left[1 - \frac{v^2}{c_s^2}\right] = -\frac{2v}{r} + \frac{vGM}{c_s^2 r^2}$$

$$\frac{dv}{dr}\frac{c_s^2}{v}\left(1 - \frac{v^2}{c_s^2}\right) = -\frac{2c_s^2}{r} + \frac{GM}{r^2}$$

$$\frac{dv}{dr}\left(\frac{c_s^2}{v}-v\right)=-\frac{2c_s^2}{r}+\frac{GM}{r^2}$$

We define the sonic point r_s, where $v=c_s$

$v=c_s$ at $\frac{2c_s^2}{r}=\frac{GM}{r^2} \Rightarrow r_s=\frac{GM}{2c_s^2}$

$$\left(\frac{v}{c_s}\right)^2-\log\left(\frac{v}{c_s}\right)^2=4\log\frac{r}{r_s}+2\frac{GM}{rc_s^2}+C$$

To evaluate the constant, we write for the sonic point $r=r_s$; $v=c_s$, which leads to $C=-3$. The wind equation is thus

$$\boxed{\left(\frac{v}{c_s}\right)^2-\log\left(\frac{v}{c_s}\right)^2=4\log\frac{r}{r_s}+2\frac{GM}{rc_s^2}-3}$$

Show graphic

I - II (doubled-valued, unphysical)

III - supersonic

IV - subsonic

V and VI -> transonic

wind: subsonic from solar surface; accelerates to sonic.

Solution V is appropriate.

VI is accretion.

accretion is a "negative wind".

Mass loss

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v_r\right) = 0 \qquad (\text{Continuity})$$

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{2}{r} - \frac{1}{v_r}\frac{dv_r}{dr}$$

$$r^2 \rho(r) v_r(r) = C_1$$

(Momentum)

$$v_r \frac{dv_r}{dr} = -\frac{GM_\odot}{r^2} - \frac{1}{\rho}\frac{dp}{dr}$$

Integrate this

$$\frac{v_r^2}{2} - \frac{GM_\odot}{r} + \int \frac{dp}{\rho} = E \qquad \int F\,dr = U$$

For isothermal wind

$$\frac{v_r^2}{2} - \frac{GM_\odot}{r} + c_s^2\left(\ln\rho - \ln\rho_0\right) = E$$

$$\text{at } r=0\ ; \quad E = \frac{v_r^2(r_0)}{2} - \frac{GM_\odot}{r_0}$$

$$\rho(r) = \rho_0 \exp\left(\underbrace{-\frac{GM_\odot}{r_0 c_s^2}\left(1 - \frac{r_0}{r}\right)}_{\text{usual static/isothermal}} - \underbrace{\left(\frac{v_r^2 - v_{r_0}^2}{2c_s^2}\right)}_{\text{outflow}} \right)$$

if $v_r(r_0) \ll c_s^2$;

$$\rho(r) = \rho_0 \exp\left(-\frac{2r_s}{r_0}\left(1 - \frac{r_0}{r}\right) - \frac{v_r^2}{2c_s^2} \right)$$

$$-\frac{2r_s}{r_0}\left(1 - \frac{r_0}{r_s}\right) \to -\frac{2r_s}{r_0} + 2 \qquad - \frac{c_s^2}{2c_s^2}$$

$$\rho(r_s) = \rho_0 \exp\left(-\frac{2r_s}{r_0} + \frac{3}{2} \right)$$

The mass loss is

$$\frac{dM}{dt} = 4\pi r_0^2 \rho(r_0) v_r(r_0) = 4\pi r_s^2 \rho(r_s) c_s$$

$$= \pi \frac{(GM_\odot)^2}{c_s^2} \rho(r_s) \approx 10^{-14} \frac{M_\odot}{yr}.$$

wind with rotation

$$v_r \frac{dv_r}{r} - \frac{v_\phi^2}{r} = -\frac{GM_\odot}{r^2} - \frac{1}{\rho}\frac{dp}{dr}$$

$$v_r \frac{dv_\phi}{dr} + \frac{v_r v_\phi}{r} = 0$$

$$r v_\phi = L = r_0^2 \Omega_\odot \qquad \therefore \; v_\phi = \frac{L}{r} = \frac{r_0^2 \Omega_\odot}{r}$$

$$v_r \frac{dv_r}{dr} - \frac{L^2}{r^2} + \frac{GM_\odot}{r^2} + \frac{1}{\rho}\frac{dp}{dr} = 0$$

$$\therefore \int \rightarrow \quad \frac{v_r^2 + v_\phi^2}{2} - \frac{GM_\odot}{r} + \int \frac{dp}{\rho} = E$$

$$\rightarrow \frac{v_r^2 + v_\phi^2}{2} - \frac{GM_\odot}{r} + c_s^2 \ln \frac{\rho(r)}{\rho_0} = E$$

+ Continuity:

$$\frac{1}{v_r}\left(\frac{v_r^2}{c_s^2} - 1\right)\frac{dv_r}{dr} = \frac{2}{r}\left(1 - \frac{GM_\odot}{2c_s^2 r} + \frac{L^2}{2c_s^2 r^2}\right)$$

Can write that as $\tau = \frac{L}{r_0 c_s} = \frac{r_0 \Omega_\odot}{c_s}$

Term is $\frac{\tau^2}{2}\left(\frac{r_0}{r}\right)^2$

$\tau_\odot \simeq 0.01$

$\Omega_\odot \approx 2.8 \times 10^{-6}\, s^{-1}$ $R_\odot \Omega_\odot \approx 2 \times 10^{5}$ cm/s
$r_\odot \approx 12 R_\odot$ $T \sim 1\text{-}2 \cdot 10^6 K \rightarrow \tau_\odot \approx 0.01$

$$\frac{v_r^2}{c_s^2} - \ln\left(\frac{v_r^2}{c_s^2}\right) = 4 \ln\left(\frac{r}{r_s}\right) - 4\frac{r_s}{r} + \tau^2 \frac{r_o^2}{r^2} + C$$

At $r = r_s$; $v_r = c_s$

$$C = -3 - \tau \frac{r_o^2}{r_s^2}$$

$$\frac{v_r^2}{c_s^2} - \ln\left(\frac{v_r^2}{c_s^2}\right) = 4 \ln\left(\frac{r}{r_s}\right) - 4\frac{r_s}{r} - \tau^2 \frac{r_o^2}{r_s^2}\left(\frac{r_s}{r} - 1\right)^2$$

Rotating magnetized wind

$\nabla \cdot B = 0 \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 B_r\right) = 0$

$$B_r = B_o \left(\frac{r_p}{r}\right)^2$$

$$\nabla\times(v\times B)=0$$

$$\frac{1}{r}\frac{d}{dr}\Big[r(v\times B)_\theta\Big]\hat{\phi}$$

$$v_\phi B_r - v_r B_\phi = \frac{C_3}{r}$$

Determine C_3 by using values at r_0:

$$\underbrace{v_\phi(r_0)}_{\Omega_0 r_0}\underbrace{B_r(r_0)}_{B_0} - \underbrace{v_r(r_0)\,B_\phi(r_0)}_{\approx 0} = \frac{C_3}{r_0}$$

$$C_3=\Omega_0 r_0^2 B_0 \quad \therefore \quad \frac{C_3}{r}=\Omega_0\frac{r_0^2B_0}{r}=\Omega_0 r B_r$$

$$\therefore B_r(v_\phi - r\Omega_0) - v_r B_\phi = 0$$

At large distances:

$$v_\phi\approx 0 \;;\; v_r \equiv V \equiv cte$$

$$-rB_r\Omega_0 = v_r B_\phi \rightarrow \frac{B_r}{B_\phi} = -\frac{V}{r\Omega_0}$$

From flux-freezing, B and streamlines coincide, so $\vec{B} \parallel d\vec{\ell}$; or $\vec{B} \times d\vec{\ell} = 0$. So:

$$B \times d\ell = B_r \, r \, d\phi - dr \, B_\phi = 0$$

$$\frac{B_r}{B_\phi} = \frac{1}{r}\frac{dr}{d\phi} \quad \therefore \quad \frac{dr}{d\phi} = -\frac{V}{\Omega_\odot}$$

$$dr = -\frac{V}{\Omega_\odot}\, d\phi$$

Thus, the wind follows

$$\boxed{r = r_* - \frac{V}{\Omega_\odot}(\phi - \phi_*)}$$

This is the equation of an Archimedian spiral!