#2

Magneto Hydro Dynamics (MHD)

Selected Topics in Astrophysics

Alfvén Flux-Freezing theorem

As we saw last class, the magnetic field evolution is given by the induction equation

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B$$

The range of human experience is often given by the latter term only, which is resistivity (diffusion of magnetic field). Astrophysical systems are often in the opposite end, of high magnetic Reynolds numbers, so that

$$\frac{\partial B}{\partial t} \simeq \nabla \times (v \times B).$$

One of the consequences is that the field has inertia. It is frozen in the plasma and follows streamlines. This is the flux-freezing theorem of Hans Alfvén, that we will explore next:

The magnetic flux is $\Phi = \int B \cdot dS$, where dS is an area element. The theorem states that

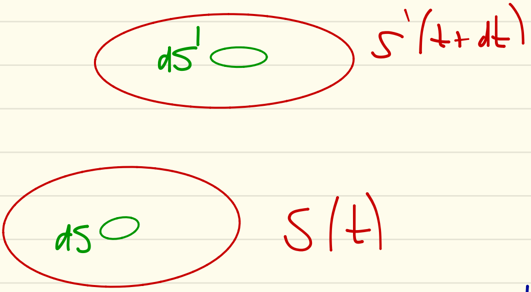
$$\frac{d\Phi}{dt} = 0$$

That is,

$$\frac{d}{dt}\int B \cdot dS = 0$$

$$\Rightarrow \int \frac{\partial B}{\partial t} \cdot dS + B \cdot \frac{d}{dt}(dS).$$

The term $\frac{d}{dt}(dS)$ is the time-variation of a surface element dS

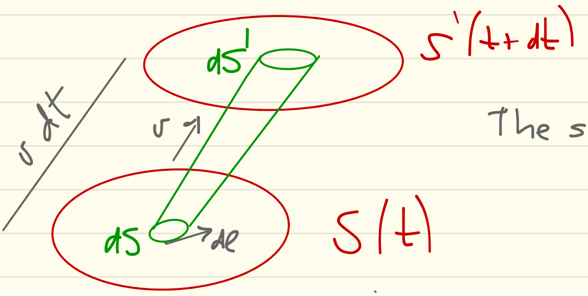


S in time t
Goes to S in time t+dt

$$\frac{d}{dt}(dS) = \lim_{\Delta t \to 0} \frac{dS' - dS}{\Delta t}$$

How do we find $dS' - dS$?

The vector area integral over a closed surface is zero: $\oint dS = 0$
(give cube as example). So, let's close the surface.



The side area is $dA = dl \times v\,dt$.

Area:

$$(dS' - dS)\,\hat{n}_1 + \oint dl \times v\,dt = 0$$

$$dS' - dS = dt \oint v \times dl = 0$$

$$\frac{d}{dt}\left(dS' - dS\right) = \lim_{\Delta t \to 0}\left(\frac{\cancel{dt}\oint v \times dl}{\cancel{\Delta t}}\right) = \oint v \times dl$$

$$\int_S B \cdot \frac{d(dS)}{dt} = \int_S B \cdot \left(\oint v \times dl\right) = \int_S \oint B \cdot v \times dl$$

$$= \int_S \oint (B \times v) \cdot dl$$

this last one, $B \cdot (v \times dl) = (B \times v) \cdot dl$, comes from

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

So

$$\int_S B \, \frac{d(dS)}{dt} = \int_S \oint (B \times v)\, dl$$

This is a strange integral. It represents the sum over all little $d\vec{l}$ elements that are embedded by the area S. We need to do line integrals over all surface elements that compose S. Because the internal line integrals cancel out (show example of honeycomb pattern) this reduces to an integral over the contour C of S.

$$\int_S \oint (B \times v) \cdot dl = \oint_C (B \times v)\, dl$$

which, by Stokes theorem, is

$$\oint_C (B \times v)\, dl = \int_S \nabla \times (B \times v)\, dS$$

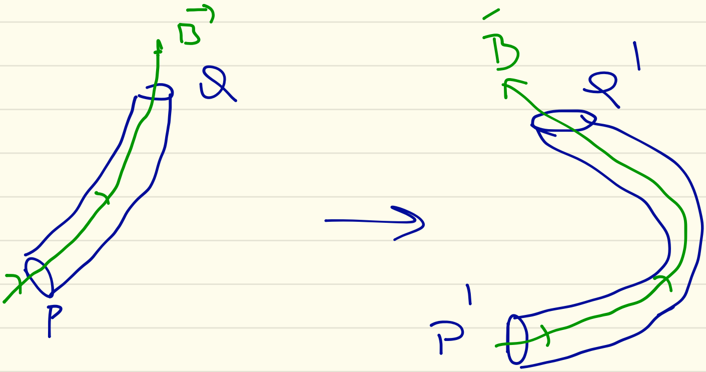
So,
$$\int_S B \frac{d(dS)}{dt} = -\int_S \nabla\times(v\times B)\, dS$$

and the quantity we were trying to find is the surface integral of the RHS of the induction equation, the electromotive force.

So,
$$\frac{d}{dt}\int_S B\, dS = \frac{d\Phi}{dt} \quad \therefore \quad \boxed{\frac{d\Phi}{dt} = \int\left[\frac{\partial B}{\partial t} - \nabla\times(v\times B)\right]dS = 0}$$

Q.E.D. for Alfvén's flux-freezing theorem.

Interpretation: Once two surface elements are connected by a field line, they will remain so. The field is frozen-in and moves with the plasma.



The B-field follows streamlines.

The surface elements P and Q change into P' and Q' as a result of fluid motions. Due to flux-freezing, the magnetic field line that connected PQ will also connect P'Q' and the field topology is preserved. This gives a neat visualization of the field, as we can guess its configuration if we know the fluid flow. The magnetic field is an almost material, plastic substance that moves with the flow.

Consequence: field in pulsars

$$\frac{d}{dt}\int B\, dA = 0$$

Sun ~ 1 G
Pulsar $\sim 10^{10}$ G

Gravitational collapse

Sun $\to 10^{11}$ cm $B = 10$ G

Neutron star $\approx 10^{6}$ cm

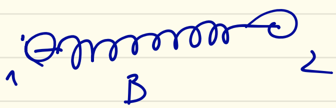
10^{10} decrease in equatorial area

10 G $\to 10^{11}$ G (fields of neutron stars are of this order).

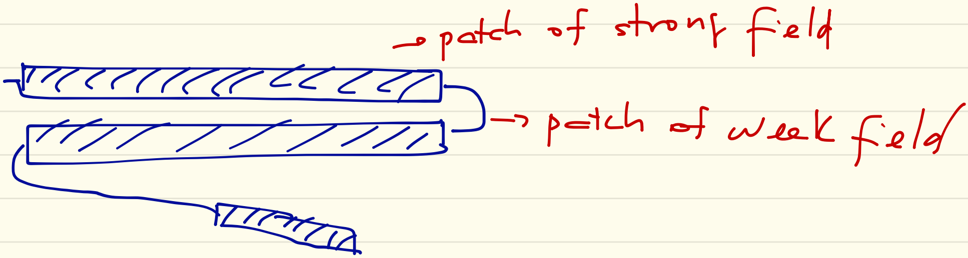
Tension

Mag fields have tension, $(B \cdot \nabla) B$, which is resistance to stretching.

Think of fields as springs connecting fluid elements



Since this is proportional to B^2, strong fields have more tension than weaker fields. Think of harder and looser springs.



Weak fields are ductile like twine. Strong fields are stiff.

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) = \underbrace{-(u \cdot \nabla) B}_{\text{advection}} + \underbrace{(B \cdot \nabla) u}_{\text{stretching}} - \underbrace{B(\nabla \cdot u)}_{\text{compression}}$$

Alfvén velocity

The plasma allows for different osillations than just sound waves. To derive them, let us consider the MHD equations

$$\frac{\partial \rho}{\partial t} = -(u \cdot \nabla)\rho - \rho(\nabla \cdot u)$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho}\nabla p + \frac{(\nabla \times B) \times B}{4\pi\rho}$$

$$\frac{\partial B}{\partial t} = -(u \cdot \nabla) B + (B \cdot \nabla) u - B(\nabla \cdot u)$$

with isothermal equation of state.

$$p = \rho c_s^2$$

Now let us make the approximation that the system is magnetically dominated ($p \ll B^2$, so $p, \nabla p \approx 0$), and that the gas is incompressible ($\nabla \cdot u = 0$). The system reduces to

$$\frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla) u + \frac{(B \cdot \nabla) B}{4\pi\rho} - \frac{\nabla B^2}{8\pi\rho}$$

$$\frac{\partial B}{\partial t} = -(u \cdot \nabla) B + (B \cdot \nabla) u - B(\nabla \cdot u)$$

Now decompose the system into base state and fluctuation

$$u = u_0 + u' \quad ; \quad B = B_0 + B'$$

$$\frac{\partial u'}{\partial t} = -\frac{\nabla(B_0 \cdot B')}{4\pi\rho_0} + \frac{(B_0 \cdot \nabla) B'}{4\pi\rho_0}$$

$$\frac{\partial B'}{\partial t} = (B_0 \cdot \nabla) u'$$

Now we set $B_0 = B_{0x}$, i.e., the field is axial. The equation for u_x' cancels out. As a consequence so does the one for B_x'. We are left with eqs for u_y and B_y.
We take the time-derivative of both equations to find the wave eqs:

$$\frac{\partial^2 u'}{\partial t^2} = \frac{B_0^2}{4\pi\rho_0} \nabla^2 u' \qquad \frac{\partial^2 B'}{\partial t^2} = \frac{B_0^2}{4\pi\rho_0} \nabla^2 B'$$

These are equations that describe waves propagating with velocity

$$v_A = \frac{B}{\sqrt{4\pi\rho}}$$

These waves are called Alfvén waves, and the associated velocity the Alfvén velocity. So:

$$\frac{\partial^2 u}{\partial t} = v_A^2 \nabla^2 u \quad ; \quad \frac{\partial^2 B}{\partial t} = v_A^2 \nabla^2 B$$

If you perturb a magnetic field, waves will propagate transversally to it, with tension as the restoring force. These waves are akin to the waves produced when plucking a guitar string.