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Magneto Hydro Dynamics (MHD)

Selected Topics in Astrophysics

There are two main theories in astrophysics that account for many of the phenomena in the Universe. One we have already covered in detail, that of Radiative Transfer, that describes radiation in the continuum, macroscopic, limit, and is paramount to interpret observations and understand the structure of stars, for which the radiation field cannot be ignored. Another equally important theory for understanding the structure of some astrophysical objects is Magnetohydrodynamics (MHD for short), that deals with the effects of the ubiquitous magnetic field in the Universe. Electromagnetic fields obey Maxwell's equations. In Gaussian units, they read

$$\nabla \cdot E = 4\pi\rho$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t}$$

And a particle or gas parcel will of course be subject to the Lorentz force

$$F_L = q\left(E + \frac{v}{c} \times B\right)$$

Let us see how these equations can be properly transported to astrophysicals environments. But first, a primer on EM in Gaussian (cgs) units. The main difference is in the definition of charge. In SI units, the Coulomb law is

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \quad (SI)$$

whereas in cgs, the constant is absorbed in the unit of charge.

$$F = \frac{q_1 q_2}{r^2} \hat{r} \quad (cgs)$$

The unit of charge in cgs is the "electrostatic unit" (esu):

$$esu = (dyne \cdot cm^2)^{1/2} \rightarrow (dyne)^{1/2} \cdot cm$$

In practice, the conversion is done by setting $\varepsilon_0 \rightarrow \frac{1}{4\pi}$

In these units then, the energy in an electric field passes from

$$U = \frac{\varepsilon_0}{2} \int E^2 dV \; (SI) \longrightarrow U = \frac{1}{8\pi} \int E^2 dV \; (cgs)$$

Also, for Biot-Savart law:

$$B = \frac{\mu_0}{4\pi} I \int \frac{dl \times \hat{r}}{r^2} \; (SI) \rightarrow B = \frac{I}{c} \int \frac{dl \times \hat{r}}{r^2} \; (cgs)$$

And the Lorentz force

$$F = q(E + v \times B) \; (SI) \rightarrow F = q\left(E + \frac{v}{c} \times B\right) \; (cgs)$$

The magnetic field is "scaled up" by c.

The neat thing about these units is that the electric and magnetic fields have the same unit. The total energy in SI and cgs are

$$U = \frac{1}{2}\int \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \, dV \quad \text{(SI)} \quad \longrightarrow \quad U = \frac{1}{8\pi}\int E^2 + B^2 \, dV \quad \text{(cgs)}$$

where we see that in cgs we get rid of the constants that spoil the symmetry in SI.

Local neutrality

A plasma is locally ionized but globally neutral. That means that electric fields can be ignored. Down to what scales is it reasonable to ignore the electric field? To answer that we compute the Debye length. Consider a species is ionized giving out Z electrons, leaving the ion with charge Ze. If n_i is the ion density and the electron density is n_e, the charge density is $e(Zn_i - n_e)$. For charge neutrality, $n_e = Zn_i$. The Poisson equation for the electrostatic potential is

$$\nabla^2 V = -4\pi (Zn_i - n_e) e$$

If the system is in thermodynamical equilibrium, we can write

$$n_i = \bar{n}_i \exp\left(-\frac{ZeV}{k_B T}\right) \qquad n_e = \bar{n}_e \exp\left(\frac{eV}{k_B T}\right)$$

where bar means average concentration. The Poisson equation becomes

$$\nabla^2 V = 4\pi \left(\bar{n}_e \exp\left(\frac{eV}{k_B T}\right) - Z\bar{n}_i \exp\left(-\frac{ZeV}{k_B T}\right)\right) e$$

For charge neutrality $\bar{n}_e = Z\bar{n}_i$ and thus

$$\nabla^2 V = 4\pi Z\bar{n}_i \left(\exp\left(\frac{eV}{k_B T}\right) - \exp\left(-\frac{ZeV}{k_B T}\right)\right) e$$

And assuming $eV \ll k_B T$,

$$\nabla^2 V = -\frac{4\pi \bar{n}_i e^2}{k_B T} Z(Z+1) V$$

And we can define the Debye length

$$\lambda_D = \left(\frac{k_B T}{8\pi \bar{n}_i e^2} \frac{1}{Z(Z+1)}\right)^{1/2}.$$

And $V = V_0 e^{-x/\lambda_D}$. That is, the potential and the field decrease exponentially, with e-folding distance equal to the Debye length λ_D. This distance is the distance up to which electric fields are relevant. For 300 K and $\bar{n}_i = 100\ \mathrm{cm}^{-3}$, $\lambda_D \simeq 10$ cm.

Lorentz transformation

We divide the electromagnetic fields into components, parallel and perpendicular to motion. These components transform according to

$$E'_{\parallel} = E_{\parallel} \qquad B'_{\parallel} = B_{\parallel}$$

$$E'_{\perp} = \gamma\left(E_{\perp} + \frac{v}{c} \times B_{\perp}\right) \qquad B'_{\perp} = \gamma\left(B_{\perp} - \frac{v}{c} \times E_{\perp}\right)$$

where prime means the field in the reference frame of motion. If the motion is non-relativistic, $\gamma \approx 1$, and thus

$$E' = E + \frac{v}{c} \times B$$

According to Ohm's law

$$J = \sigma E'$$

or

$$J = \sigma\left(E + \frac{v}{c} \times B\right)$$

If $\sigma \to \infty$ (a highly conductive medium), then $E' \to 0$

that is, $E \approx \frac{|v|}{c} B$

The electric field in the reference frame of motion is zero, and even E is of order v/c, which will be small for non-relativistic motion. This is in line with what we expect from charge neutrality. As for the magnetic field

$$B' \to B - \frac{v}{c} \times E \to B + O(v^2/c^2) \approx B$$

And Ampère's law

$$\nabla \times B = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

Let us compare the magnitude of the displacement current with the LHS.

$$\frac{1/c\ \partial E/\partial t}{|\nabla \times B|} \approx \frac{|E|/ct}{|B|/\ell} \approx \frac{|v|}{c}\frac{|E|}{|B|}$$

$$\text{if } |E| \approx \frac{|v|}{c}|B| \quad \therefore \quad \frac{1/c\ \partial E/\partial t}{|\nabla \times B|} \approx \frac{v^2}{c^2}$$

I.e., the displacement current is of order v^2/c^2 and can be ignored. Thus Ampère's law becomes

$$\nabla \times B = \frac{4\pi J}{c}$$

For the electric field, we substitute J as given by Ohm's law

$$E = \frac{c}{4\pi\sigma} \nabla \times B - \frac{v}{c} \times B$$

which results in a term proportional to $1/r \to 0$; and a second one, proportional to v/c. Thus, the electric field is always negligible in the plasma frame. Also, given by the expression above, the electric field is not an independent variable. It can always be calculated knowing v and B

If we can ignore E; then

$$F_{LORENTZ} = q\left(E + \frac{v}{c}\times B\right) = \overset{J}{\frac{q\,v}{c}}\times B = \frac{J\times B}{c}$$

And the Navier-stokes equation with the Lorenz force i

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{\rho}\nabla p + \frac{J\times B}{\rho c} + \nu\nabla^2 u$$

with the current given by

$$J = \frac{c}{4\pi}\nabla\times B$$

We can then write

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}(\nabla\times B)\times B + \nu\nabla^2 v$$

And using the following vector identity

$$(\nabla\times B)\times B = (B\cdot\nabla)B - \nabla\left(\frac{B^2}{2}\right)$$

So,
$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B}{4\pi\rho} + \nu\nabla^2 v$$

The term $B^2/8\pi$, that enters as a gradient and behaves like a pressure term <u>is</u> a pressure term, the magnetic pressure.

The term $(B\cdot\nabla)B$ is the magnetic tension.

Magnetic Tension

Magnetic tension is the non-isotropic part of the Lorentz force. It can be interpreted as the resistance the field offers to being bent. Let us write the Lorentz force in component notation:

$$\frac{1}{4\pi\rho}\left[(\nabla\times B)\times B\right]_i = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\frac{B^2}{8\pi}\delta_{ij} - \frac{B_iB_j}{4\pi}\right]$$

The RHS is the divergent of the Maxwell tensor

$$M_{ij} = \frac{B^2}{8\pi}\delta_{ij} - \frac{B_iB_j}{4\pi}$$

And we can write Euler's Eq. as

$$\rho\frac{dv_i}{dt} = \rho F_i - \frac{\partial}{\partial x_j}\left(P_{ij} + M_{ij}\right)$$

Let us illustrate the tension with the case $B = B\hat{z}$. Then

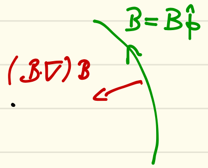
$$M_{ij} = \begin{bmatrix} B^2/8\pi & 0 & 0 \\ 0 & B^2/8\pi & 0 \\ 0 & 0 & -B^2/8\pi \end{bmatrix}$$

The tensor is isotropic aside from the M_{zz} term. To restore symmetry we can write

$$M_{zz} = \frac{B^2}{8\pi} - \frac{B^2}{4\pi}$$

where the 1st term combines with M_{xx} and M_{yy} for the isotropic pressure. The 2nd term is the tension. In this case, it behaves like negative pressure.

Another illustrative case is for an azimuthal field $B = B\hat{\phi}$. In this case, $(B\cdot\nabla)B = -\frac{B^2}{r}\hat{r}$.



That is, if you bend a field azimuthally, it will give rise to a centripetal force. It's like trying to bend a bar: it will offer resistance, trying to straighten the field to a force-free configuration

Induction equation

According to Faraday's law

$$\frac{\partial B}{\partial t} = -c\,\nabla\times E$$

with the electric field given by

$$E = \frac{c}{4\pi\sigma}\nabla\times B - \frac{v}{c}\times B$$

Becomes:

$$\frac{\partial B}{\partial t} = -c\,\nabla\times\left[\frac{c}{4\pi\sigma}\nabla\times B - \frac{v}{c}\times B\right]$$

$$= \nabla\times\left(v\times B - \frac{c^2}{4\pi\sigma}\nabla\times\nabla\times B\right)$$

Replace the double curl by the following identity

$$\nabla\times\nabla\times B = \cancel{\nabla(\nabla\cdot B)}^{\,0} - \nabla^2 B$$

where $\nabla\cdot B = 0$ because of Gauss' law. The induction equation then is

$$\boxed{\frac{\partial B}{\partial t} = \nabla\times(v\times B) + \eta\,\nabla^2 B}$$

where $\eta = c^2/4\pi\sigma$ is the resistivity.

We can construct a dimensionless number, Re_m, the magnetic Reynolds number, that gives the relative importance of the first and second terms in the RHS.

$$Re_m = \frac{\nabla\times(v\times B)}{\eta\,\nabla^2 B}$$

And we have two regimes, based on the Reynolds number

If $R_m \ll 1$; $\dfrac{\partial B}{\partial t} = \eta\,\nabla^2 B$

If $R_m \gg 1$; $\frac{\partial B}{\partial t} = \nabla \times (u \times B)$

The former is a simple diffusion equation. In the latter, the RHS is the electromotive force.

In dimensions, Re_M can be written as

$$[Re_m] = \frac{1/L \cdot U}{\eta \, 1/L^2} = \frac{LU}{\eta}$$

Let us estimate Re_m on human scales and in astrophysics. In the lab: $L \simeq 10^2$ cm and $U \approx 10$ cm/s. For the Sun, $L \approx 10^8$ cm, and $U \approx 10^5$ cm/s.

A typical resistivity is of order 10^7 cm/s

So, in the lab, $Re_M = \frac{10^2 \cdot 10}{10^7} \approx 10^{-4}$

In space, $Re_M = \frac{10^8 \cdot 10^5}{10^7} \approx 10^6$

So, in human experience, we are deep within the resistive behaviour of the field. Indeed, in the lab a field is maintained only as long as a current is applied. In space the behaviour of the magnetic field is completely different dominated by the electromotive force. As we will see, among other things the field has inertia, and tension. These are features of the magnetic that we have no intuition for from our experience. In the next classes we will build intuition for the magnetic field in astrophysics.