Planetary Sciences #8

class 19

Interiors of giant planets

Giant planets are in hydrostatic equilibrium, so we can use the same equations from stellar structure to describe them

$$\frac{dP(r)}{dr} = -\frac{GM_r}{r^2}\rho$$

It is noticeable that Jupiter and Saturn have approximately the same size, yet their masses differ by almost a factor 3. We have derived in homework assignment #4 the solution of the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

(where $r = \alpha\xi$ and $\rho = \rho_c\theta^n$) for a polytrope of index $n=1$

i.e. $P = K\rho^{1+1/n} = K\rho^2$.

The solution was $\theta = \frac{\sin\xi}{\xi}$. That means $\rho = \rho_c\frac{\sin\xi}{\xi}$. The expression for α is $\alpha^2 = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}$, which for $n=1$ reduces to

$\alpha = \left(\frac{K}{2\pi G}\right)^{1/2}$,

The radius of a body is found by setting the upper boundary at $\rho = 0$.

That is

$$\rho = \rho_c\frac{\sin\xi}{\xi} = 0 \quad \therefore \sin\xi = 0 \quad \rightarrow \quad \xi = \frac{r}{\alpha} = \pi$$

$$\boxed{R = \left(\frac{K\pi}{2G}\right)^{1/2}}$$

This solution for the radius of a n=1 polytrope is quite peculiar: it does not depend on the mass. Plugging in the value of K:

$$K \simeq 2.7 \cdot 10^{12} \ \frac{\text{dyne} \cdot \text{cm}^4}{g^2},$$

we find

$$\boxed{R \simeq 8 \times 10^4 \ \text{Km}}$$

This is the only possible radius for this polytrope. The radii of Jupiter and Saturn are

$$\text{Jup} \simeq 69.911 \ \text{Km}$$
$$\text{Sat} \simeq 58.232 \ \text{Km}$$

which strongly hints they are reasonably-well described by this polytrope. Gas giant planets and brown dwarfs are polytropes of index n=1.

Degeneracy parameter

$$\theta = \frac{E_T}{E_F} = \frac{2m_e k}{h^2}\left(\frac{8\pi}{3}\mu_e m_u\right)^{2/3} \frac{T}{\rho^{2/3}}$$

If $\theta<1$, electron momentum is not determined by temperature, but due to the fact that you can only stack two electrons (+1/2 − 1/2) in a volume $\Delta p \Delta V = h^3$

Substellar objects always have $\theta < 1$.

Coulomb-to-thermal energy

$$\Gamma = \frac{e^2}{aKT}$$ a is the mean distance between nuclei

$$= \frac{e^2}{K}\left(\frac{4\pi}{3\mu m_H}\right)^{1/3} \frac{\rho^{1/3}}{T}$$

If $\Gamma > 1$, the Coulomb force is more effective. The ions get bound into lattice for $\Gamma \gg 1$.

In planets, $\Gamma \simeq 50$, too low for crystallization, which happens at $\Gamma \simeq 180$.

Pressure ionization

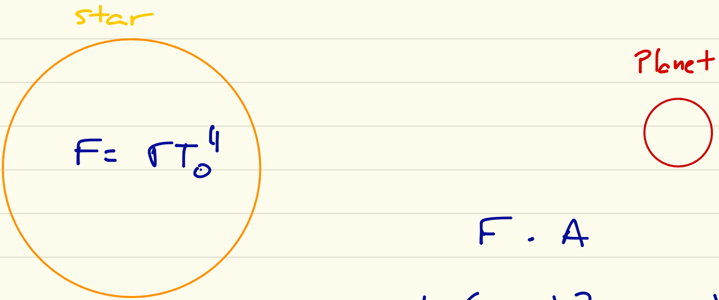
We are used to ionization at high temperature, but high pressures can also lead to ionization. High pressure ionization occurs when the Fermi energy is larger than that needed to ionize hydrogen ($\chi_0 = 13.6$ eV).

$$n_e \gtrsim \left(\frac{8\pi}{3}\right)^{5/2} \left(\frac{5 m_e}{4\pi h^2}\right)^{3/2} \chi_0^{3/2}$$

$$n_{e\,crit} \approx 5 \times 10^{23}\ cm^{-3} ; \rho \approx 0.8\ g/cm^{-3}$$

Which is about the density of saturn.
The critical P_e (ionization pressure) is about 7 Mbar.

Planet's Equilibrium Temperature



$F \cdot A$ cross section

Receiving Energy $\sigma T_\odot^4 \left(\frac{R_\odot}{D}\right)^2 (1-a) \cdot \pi R^2$

Emitted Energy $\sigma T_p^4 \cdot 4\pi R^2$

$$T_p^4 = T_\odot^4 \left(\frac{R_\odot}{D}\right)^2 \left(\frac{1-a}{4}\right)$$

$$\boxed{T_p = T_\odot (1-a)^{1/4} \sqrt{\frac{R_\odot}{2D}}}$$

a is albedo
D: star-planet distance

Does not depend on planet radius because incoming and outgoing radiation both depend on the planet's area.

For Earth, this is $-18^\circ C$. ($\approx -1^\circ F$).

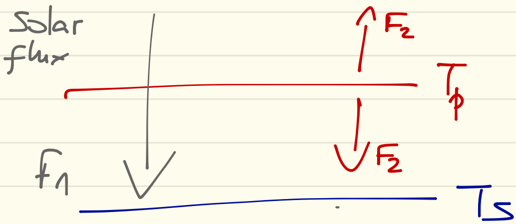
The Earth is significantly warmer than this ($\approx 15^\circ C$), so why the discrepancy?

Greenhouse

If an atmosphere is opaque in infrared, short wavelengths can go in, but long wavelengths cannot get out.

Let us model this in an ideal situation. Imagine a single layer atmosphere that is completely transparent in visible and opaque in infrared. What is the surface temperature?

Geometry of the problem:



The top of the atmosphere is heated by the sun alone in this approximation, so it heats up to the equilibrium temperature. This layer will radiate this flux isotropically, so the surface is heated by the sum of fluxes F_1, from the star, and F_2, from the atmosphere.

$$F = F_1 + F_2$$

$$\sigma T_{surf}^4 = F_1 + \sigma T_p^4$$

F_1 is equal to $\sigma T_\odot^4 (1-a) R^2/4D^2$, but we can also write is as σT_p^4, since this is the flux that heats up a planet to its equilibrium temperature. So,

$$T_{surf}^4 = 2\, T_p^4 \quad \rightarrow \quad T_{surf} = 2^{1/4}\, T_p$$

For $T_p = 255$, $T_{surf} \approx 300$ K.

OK, that was closer to Earth's temperature, but the approximations were very coarse. Let's look at it in more detail. The atmosphere in infrared is heated from below, so we can use the theory of radiative transfer developed for stars. In the two-stream, grey approximation in radiative equilibrium, the temperature as a function of depth is

$$T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3}\right) = \frac{T_{eff}^4}{2} \left(1 + \frac{3\tau}{2}\right)$$

And we need to find the "effective temperature" of the atmosphere. This is not the equilibrium temperature T_0 at the top of the atmosphere for the same reason that in stars the effective temperature is not the surface temperature: we see photons coming from hotter layers below. The temperature at the top of the atmosphere ($\tau = 0$) is

$$T_0^4 = T_{eff}^4 / 2 \quad \Rightarrow \quad T_{eff} = 2^{1/4} T_0$$

So,

$$\boxed{T^4 = T_0^4 \left(1 + \frac{3\tau}{2}\right)}$$

This is the greenhouse-corrected temperature. For Earth, $\tau \simeq 1$, so the temperature rises from $T_0 = 255K$ to $T = 300K$. For Venus, $\tau \simeq 70$ so the temperature passes from $T_0 = 260K$ to $T = 750K$.