Planetary Sciences #5

STA #16

Show HR 8799 as possible evidence for GI.

Show distribution of exoplanets: giant planets in tight orbits as evidence for:

Disk Migration

$J = m\, r \times v$

if circular orbit; $r = a \equiv$ SMA ; $v = \Omega a$

$J = m \Omega a^2$; $\Omega = (GM)^{1/2} \cdot a^{-3/2}$

$\therefore \quad J = m (GM)^{1/2} a^{1/2}$

$$dJ = \frac{m (GM^{1/2})}{2a^{1/2}}\, da = \frac{m \Omega a}{2}\, da$$

$$\frac{dJ}{dt} = \Gamma = \frac{m \Omega a}{2} \frac{da}{dt} \Rightarrow \boxed{\frac{da}{dt} = \frac{2\Gamma}{m \Omega a}}$$

The sign of the torque determines if a planet migrates inward (negative torque) or outward (positive torque).

Simple analysis. What is the velocity gained by a gas parcel or particle as it approaches a planet?

Impulse approximation.

$$F_\perp = m\frac{dv_\perp}{dt} \Rightarrow v_\perp = \int_{-\infty}^{\infty} dv_\perp = \frac{1}{m}\int_{-\infty}^{\infty} F_\perp\, dt$$

from the geometry

$$F_\perp = F\sin\theta = \frac{F\cdot b}{r} = \frac{GMm}{r^2}\frac{b}{r^3}$$

Lets approximate $v_{init} \approx v_{final} \approx v$

$$v_\parallel\, dt = v\, dt = ds \rightarrow dt = \frac{ds}{v}$$

$$v_\perp = \frac{1}{m}\int_{-\infty}^{\infty} F_\perp\, dt = \frac{2}{m}\int_0^{\infty}\left(\frac{GMm}{r^2}\right)\left(\frac{b}{r}\right)\frac{1}{v}\, ds$$

$$r^2 = s^2 + b^2$$

$$v_\perp = \frac{2\,GM}{v}\int_0^{\infty}\frac{b}{(s^2+b^2)^{3/2}}\, ds = \frac{2GM}{v}\int_0^{\infty}\frac{ds}{b\,(1+s^2/b^2)^{3/2}}$$

$$x = s/b$$

$$v_\perp = \frac{2GM}{vb}\int_0^{\infty}\frac{dx}{(1+x^2)^{3/2}} = \frac{2GM}{vb}\left.\frac{x}{(1+x^2)^{1/2}}\right|_0^{\infty}$$

$$v_\perp = \frac{2GM}{vb}$$

For small angle deflection $\varphi = v_\perp / v$

$$\varphi = \frac{2GM}{v^2 b}$$

$v \equiv \Delta v = v_{gas} - v_p$ $\qquad v_\perp \equiv \delta v_\perp$

$\qquad\qquad\qquad\qquad\qquad\qquad v_\parallel \equiv \delta v_\parallel$

$v_\perp$ implies no change in angular momentum

Cons of energy:

$$\Delta v^2 = (\delta v_\perp)^2 + (\Delta v - \delta v_\parallel)^2$$

(before) $\qquad\qquad\qquad$ (after)

$$\cancel{\Delta v^2} = \delta v_\perp^2 + \cancel{\Delta v^2} + \cancel{\delta v_\parallel^2} - 2\Delta v\, \delta v_\parallel$$

small deflection

$$\delta v_\parallel \simeq \frac{\delta v_\perp^2}{2\Delta v} = \frac{1}{2\Delta v}\left(\frac{2GM_p}{b\Delta v}\right)^2$$

The change in angular momentum is

$$\Delta j = r \times v = a\,\delta v_{\parallel} = \frac{2G^2 M_p^2 a}{b^2 \Delta v^3}$$

Sign $\Rightarrow$ Gas exterior to planet is overtaken by planet; $\Delta v < 0$; planet loses ang momentum

Gas interior overtakes, $\Delta v > 0$; planet gains ang momentum.

Net direction of migration depends on difference btw interior and exterior torque.

For total torque, integrate for whole disk

Mass in b; $b + db$ $\rightarrow$ $dm = 2\pi a \Sigma\, db$

gas $= \Omega$; planet Ω_p, gas suffer impulses separated by

$$\Delta t = \frac{2\pi}{|\Omega - \Omega_p|} \quad \text{(synodic periodic)}$$

if $b \ll a$; $|\Omega - \Omega_p| = \frac{d\Omega_p}{da} \cdot b = \frac{3}{2}\frac{\Omega_p}{a} b$

Total temporal change must be the integral of angular momentum transfer of all parcels per unit time

$$\left.\frac{dJ}{dt}\right|_{planet} = -\left.\int \frac{\Delta j \, dm}{\Delta t}\right|_{disk}$$

$$\Delta v \simeq |\Omega_p'| \, a b = \frac{3}{2} \Omega_p b \quad \text{(Quasi-Keplerian orbits)}$$

$$\frac{dJ}{dt} = -\int_0^{\infty} \frac{8 G^2 M_p^2 \Sigma a}{9 \Omega_p^2 b^4} \, db$$

Diverges at inner boundary. Must give minimum b_{MIN}

$$\Gamma = \frac{dJ}{dt} = -\frac{8 G^2 M_p^2 \Sigma a}{27 \Omega_p^2 b_{MIN}^3}$$

Negative

$$b_{MIN} \approx r_H$$

This is an approximation, but gives the correct scaling. Note that

- torque scales with the SURFACE DENSITY
- " " " SQUARE OF PLANET MASS

$$\tau_{MIG} = \frac{J}{\dot{J}} \propto \frac{1}{M_p}$$

Also seen from

$$\tau_{MIG} = \frac{a}{\dot{a}} = \frac{a}{-\Omega a} \frac{M_p}{\dot{J}} = \frac{1}{M_p}$$

More massive planets migrate faster

For $M_p = 1\,M_\oplus$ at $r = 5\,AU$, $\Sigma \sim 100\ g/cm^2$,

and $b_{min} \simeq r_H$

$$\tau \sim 1\,Myr$$

$$M_p = 1\,M_J \quad \sim \quad \tau \sim 2 \times 10^5\ Myr$$

This was made by following gas pockets in time, and checking how they exchange any momentum with the disk. (Impulse approximation)

Another way is to analyze how azimuthal asymmetries in the disk gravitationally pull on the planet.

Resonances

In orbital locations where the gas is in resonance with the planet, we can expect the perturbations to the disk to be largest. Let us calculate where these resonances happen.

The planet orbits with frequency

$$\Omega_p = \left(\frac{GM}{r^3}\right)^{1/2}$$

Resonances will happen at corotation

$$\Omega = \Omega_p$$

And whenever the frequency between gas and planet matches a natural frequency of the system, κ

$$m(\Omega - \Omega_p) = \pm \kappa$$

Here m is <u>NOT</u> mass, but a positive integer

$m = 1, 2, 3, 4, \ldots$

And κ, a natural frequency of the system, is the epicyclic frequency (last class).

These resonances are known as Lindblad resonances

$$m(\Omega - \Omega_p) = \pm K$$

Recalling $\Omega^2 = GM/R^3$, we solve this for radius:

$$r_- = r_p \left(\frac{m}{m-1}\right)^{-2/3}$$ Inner Lindblad resonances

$$r_+ = r_p \left(\frac{m}{m+1}\right)^{-2/3}$$ Outer Lindblad resonances

Or, more compactly

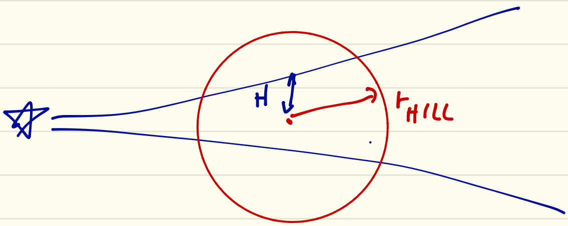
$$r_\pm = \left(1 \pm \frac{1}{m}\right)^{2/3}$$

Show slide: they form a "resonant comb", with resonances piling up near the planet as m increases.

Notice there is an asymmetry: for a given m, the outer resonance is closer to the planet. This means that the outer Lindblad spiral has a stronger effect than the inner. The outer spiral removes angular momentum, the inner spiral injects. If the outer one wins, the planet migrates inwards.

Gap opening

If a planet is massive enough, it will carve a gap in the disk. The condition, given by the geometry below



for $h=0.05$

$q \sim 10^{-4}$, or Neptune mass.

Is that the planet's Hill radius is larger than the disk scale height. If that happens, the planet disrupts the pressure equilibrium of the disk.

The scale height is given by balancing pressure and gravity:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = -\Omega^2 z \Rightarrow c_s^2 \frac{\partial \ln \rho}{\partial z} = -\Omega^2 z \Rightarrow \ln \rho = -\frac{\Omega^2 z^2}{2 c_s^2} + C$$

$$\text{So } \rho(z) = \rho_0 \, e^{-\frac{\Omega^2 z^2}{2 c_s^2}} = \rho_0 e^{-z^2/2H^2}$$

with $H = \frac{c_s}{\Omega}$ being the pressure scale heigth.

So, the condition is:

$$r_H > H$$

$$r\left(\frac{M_p}{3M_\odot}\right)^{1/3} > H$$

Call $q \equiv \frac{M_p}{M_\odot}$ the planet mass ratio. (Confusing, I know, since q was already used for the shear parameter, but alas that's the convention)

$$r\left(\frac{q}{3}\right)^{1/3} > H \implies q > 3\left(\frac{H}{r}\right)^3$$

$\frac{H}{r} = \frac{c_s}{\Omega r} = \frac{c_s}{v_K}$ is the inverse of the disk Mach number

Ma ~ 20 for disks usually. That yields

$q \sim 10^{-4}$ for gap opening, which is close to the mass of Neptune.

Show planet population synthesis model

Show observation of HL Tau