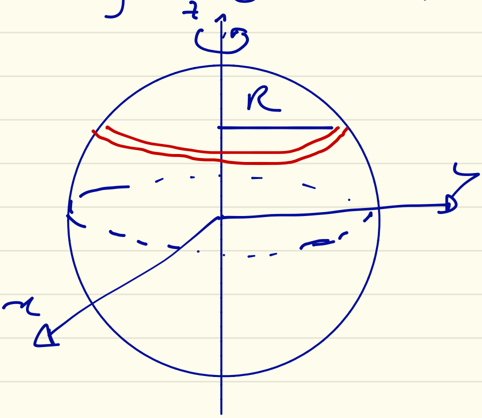
Planetary Sciences #4

STA- Class 15

A way by which we can gain knowledge about the internal composition of planets is by the inertia moment, which depends on the internal mass distribution of the planet, and is measurable. Consider the geometry below, where R is the cylindrical radius, i.e. the distance to the axis of rotation. The inertia moment is



$$I = \int \rho(r) R^2 dV$$

$$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$R = r\sin\theta$$

For constant density, the inertia moment is

$$I = 2\pi\rho \int r^4 \sin^3\theta \, d\theta \, dr \qquad I = \rho\int R^2 dV$$

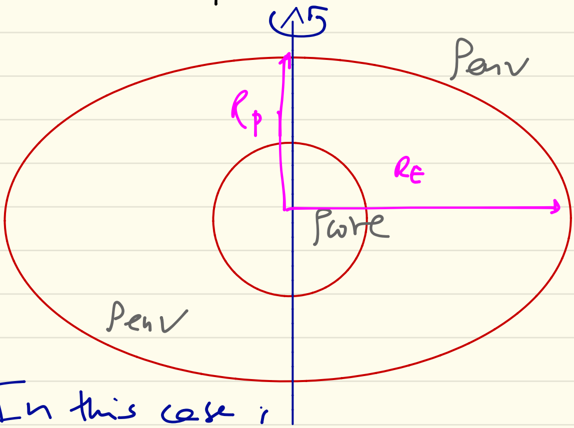
$$= 2\pi\rho \frac{r^5}{5} \int_0^\pi \sin^3\theta \, d\theta$$

$$= 2\pi\rho \cdot \frac{r^5}{5} \cdot \frac{4}{3} = \frac{8\pi}{15} \cdot \rho R^5$$

$$\rho = \frac{M}{\frac{4\pi}{3} \cdot R^3} \rightarrow I = \frac{\cancel{8\pi}^{\,2}}{\cancel{15}_{\,5}} \cdot \frac{M}{\cancel{4\pi R^3}} \cdot \cancel{3} \, R^{\cancel{5}\,2} = \frac{2}{5} M R^2$$

Deviations from constant density will lead to deviations from the 2/5 MR^2 value. Let us consider a 2-phase core + envelope model:

Core + envelope approximation



R_p: polar radius
R_e: equatorial radius

In this case:

$$I = \frac{8\pi}{15} R_e^4 \left[R_p \rho_{env} + f^5 R_e (\rho_{core} - \rho_{env}) \right]$$

$$f = \frac{R_{core}}{R_e}$$

We can conclude that I is related to oblateness and density difference between core and envelope.

More fundamentally

$$\frac{I}{MR^2} = \frac{2}{3}\left(1 - \frac{2}{5}\sqrt{\frac{5q_r}{2\varepsilon} - 1}\right)$$

where q_r is the ratio between centrifugal force and gravity, and ε is the oblateness

$$q_r = \frac{\omega_{rot}^2 R^3}{GM} \quad ; \qquad \varepsilon = \frac{R_e - R_p}{R_p}$$

Another way to measure q_r is from the gravitational moments. The deviation from sphericity can be written as

$$\Phi_g(r,\theta,\phi) = -\left(\frac{GM}{r} + \Delta\Phi_g(r,\theta,\phi)\right)$$

with

$$\Delta\Phi_g(r,\phi,\theta) = \frac{GM}{r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n\left[C_{mn}\cos(m\phi) + S_{mn}\sin(m\phi)\right]P_{nm}(\cos\theta)$$

being an expansion in spherical harmonics. We can simplify this expression by assuming azimuthal symmetry

Then $S_{nm}=0$, and for $m\neq 0$ all C_{mn} are zero. The only non-zero coefficient is C_{n0}

$$\Delta\Phi_g(r,\theta)=\frac{GM}{r}\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n C_{n0}\, P_{n0}(\cos\theta)$$

We can now define $J_n \equiv -C_{n0}$ and have

$$\boxed{\Phi_g(r,\theta,\phi) = -\frac{GM}{r}\left[1-\sum_{n=1}^{\infty} J_n\, P_n(\cos\theta)\left(\frac{R}{r}\right)^n\right]}$$

where $P_n(\cos\theta)$ are the Legendre polynomials. The first few are

n	$P_n(x)$
0	1
1	x
2	$1/2\,(3x^2+1)$
3	$1/2\,(5x^3-3x)$
4	$1/8\,(35x^4-30x^2+3)$

In the slides one sees the shapes of the J_2, J_4 and J_6 effects. The J_2 is related to pure oblateness, given by $J_2 \simeq \frac{q_r}{2}$.

Measuring I, we can plot the values (see slides) to reveal a trend: the Moon is quite close to homogeneous, and there is a downward trend to less homogeneity towards higher masses, indicating mass is getting concentrated near the center: planets have denser cores and lighter envelopes.

Jupiter is an outlier in the trend, which is odd since it is of so high mass. Indeed, the data is inconclusive over whether Jupiter has a core or not. This opens the possibility that maybe Jupiter did NOT form by core accretion, but by a top-down mechanism similar to star formation: direct gravitational instability from the gas disk.

Gravitational (disk) instability

Let us consider the equations of hydrodynamics

$$\frac{\partial \rho}{\partial t} + (v \cdot \nabla)\rho = -\rho \nabla \cdot v$$

$$\frac{\partial u}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho}\nabla p - \nabla \Phi$$

And add to it the Poisson equation for selfgravity

$$\nabla^2 \Phi = 4\pi G \rho$$

Now let's find under which conditions these three equations result in gravitational collapse. These will be some brutal pages of calculations, but bear with me because the result is worth it and we will do some interesting math in the process too

First, let us assume that the disk is infinitesimally thin, and integrate the equations in z, with $\Sigma = \int \rho \, dz$:

$$\frac{\partial \Sigma}{\partial t} + (v \cdot \nabla)\Sigma = -\Sigma \nabla \cdot v$$

$$\frac{\partial u}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\Sigma} \nabla P - \nabla \Phi$$

where $P = \Sigma c_s^2$. For the Poisson equation, we substitute $\rho = \Sigma \delta(z)$

$$\nabla^2 \Phi = 4\pi G \Sigma \delta(z)$$

Now let's expand the momentum equations in cylindrical coordinates:

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \Phi}{\partial R} - \frac{1}{\Sigma} \frac{\partial p}{\partial R}$$

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R} \frac{\partial \Phi}{\partial \phi} - \frac{1}{\Sigma R} \frac{\partial p}{\partial \phi}$$

Now let's define base state and perturbation. We decompose

$$v_\phi = \Omega R + u_\phi \, ;$$

where ΩR is the Keplerian motion (base state) and $u_\phi \ll \Omega R$ a perturbation. For the radial direction, $v_r = u_r$. So:

$$\frac{\partial u_R}{\partial t} + u_R \frac{\partial u_R}{\partial R} + \Omega \frac{\partial u_R}{\partial \phi} + \frac{u_\phi}{R}\frac{\partial u_R}{\partial \phi} - \Omega^2 R - 2\Omega u_\phi = -\frac{\partial \Phi}{\partial R} - \frac{1}{\Sigma}\frac{\partial P}{\partial R} \qquad (\mathrm{I})$$

$$\frac{\partial u_\phi}{\partial t} + u_R \frac{\partial(\Omega r + u_\phi)}{\partial R} + \left(\frac{\Omega R + u_\phi}{R}\right)\frac{\partial u_\phi}{\partial \phi} + \Omega u_r + \frac{u_\phi u_r}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{1}{\Sigma R}\frac{\partial P}{\partial \phi} \qquad (\mathrm{II})$$

The second term in eq 2 contains the shear term:

$$\frac{\partial(\Omega R)}{\partial R} = \Omega + R\frac{\partial \Omega}{\partial R} = \Omega\left(1 + \frac{\partial \ln \Omega}{\partial \ln R}\right) = \Omega(1-q)$$

where $q \equiv -\dfrac{\partial \ln \Omega}{\partial \ln R}$, from $\Omega \propto R^{-q}$ is the shear parameter.

We can then rewrite Eq II as:

$$\frac{\partial u_\phi}{\partial t} + (2-q)\Omega u_r + u_r \frac{\partial u_\phi}{\partial r} + \frac{(\Omega R + u_\phi)}{R} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{1}{\varepsilon R}\frac{\partial P}{\partial \phi} \quad \text{(II)}$$

Now let us assume a local Cartesian approximation, where x and y are given by

$$R = R_0 + x \quad ; \quad R_0 \gg x$$
$$\phi = \frac{y}{R_0} \quad ; \quad R_0 \gg y$$

where R_0 is a general location in the disk. This approximation means that we are looking at a small box inside the disk where curvature terms can be ignored. Under this approximation, the equations become

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + v_n \frac{\partial u_x}{\partial y} - \Omega^2 x - 2\Omega u_y = -\frac{\partial \Phi}{\partial x} - \frac{1}{\varepsilon}\frac{\partial p}{\partial x}$$

$$\frac{\partial u_y}{\partial t} + (2-q)\Omega u_x + v_K \frac{\partial}{\partial y} u_y + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial \Phi}{\partial y} - \frac{1}{\varepsilon}\frac{\partial P}{\partial y}$$

where $v_K = \Omega R$ is the Keplerian motion. It represents a translation of the whole box and can be removed. $v \rightarrow v - v_K$.

So, we are left with:

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma v) = 0$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{\nabla p}{\Sigma} - \nabla \phi - 2\Omega \times v + \Omega^2 \vec{r} + q\Omega v_x \hat{y}$$

$$\nabla^2 \Phi = 4\pi G \Sigma \delta(z)$$

Now let us decompose the other quantities into base state and perturbation:

$$\Sigma = \Sigma_0 + \Sigma_1(x, y, t)$$
$$v = v_1(x, y, t)$$
$$p = p_0 + p_1(x, y, t)$$
$$\Phi = \Phi_0 + \Phi_1(x, y, t)$$

Assuming again that $\psi_1 \ll \psi_0$

The unperturbed state, $\Sigma = \Sigma_0$, $\Phi = \Phi_0$; $v = 0$; $p = p_0$, yields simply

$$\nabla \Phi_0 = \Omega^2 \vec{r} = \Omega^2 (x \hat{x} + y \hat{y})$$

which is the condition of centrifugal balance.

The equation for the perturbations become

$$\frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \nabla \cdot v = 0$$

$$\frac{\partial v}{\partial t} = -\frac{c_s^2}{\Sigma_0} \nabla \Sigma_1 - \nabla \phi_1 - 2\Omega \times v + q\Omega v_x \hat{y}$$

$$\nabla^2 \Phi_1 = 4\pi G \Sigma_1 \delta(z)$$

We still have time and space derivatives that we can't yet solve. To advance, let us guess what these variations should be. We can decompose the variations in Fourier modes:

$$\psi_1 = \psi_a e^{i(kx - \omega t)}$$

whose variation in time and space is known. The equations become

$$\Sigma_1(x,y,t) = \Sigma_a e^{i(kx - \omega t)}$$

$$v_1 = (v_{ax}\hat{x} + v_{ay}\hat{y})\, e^{i(kx - \omega t)}$$

$$\Phi_1 = \Phi_a e^{i(kx - \omega t)} \quad (z \text{ plane only})$$

The last equation is only valid for $z=0$, because of the Dirac delta.

at $z \neq 0$

$$\Phi_1 = \Phi_a e^{i(kx-\omega t)} f(z)$$

And we require $\nabla^2 \Phi_1 = 0$ for $z \neq 0$

This equation gives

$$\frac{d^2 f}{dz^2} = k^2 f$$

$$\therefore \quad f = A e^{-kz} + B e^{kz}$$

Φ_1 must be finite at $z \to \pm\infty$, so $B = 0$

$$\Phi_1 = \Phi_a e^{i(kx-\omega t) - |kz|}$$

Next we need to find Σ_a, u_{xa}, u_{ya} and Φ_a. For that, we use the Poisson equation

$$\nabla^2 \Phi_1 = 4\pi G \Sigma_1 \delta(z)$$

Like the RHS, the LHS must be zero everywhere except at $z=0$. Let us integrate between $-\varepsilon$ and ε, and then take the limit $\varepsilon \to 0$

$$\int_{-\varepsilon}^{\varepsilon} \nabla^2 \Phi_1 \, dz = \int_{-\varepsilon}^{\varepsilon} 4\pi G \Sigma_1 \delta(z) \, dz$$

Take limit $\epsilon \to 0$

$$-2|k|\,\Phi_a = 4\pi G \Sigma_a$$

$$\Phi_1 = -\frac{2\pi G \Sigma_a}{|k|}\, e^{i(kx-\omega t)-|k|z}$$

The equation above allows us to substitute Φ_1 in the momentum equation. The resulting equations are

$$-i\omega\, \Sigma_a = -ik\Sigma_0 v_{ax}$$

$$-i\omega\, v_{ax} = -\frac{c_s^2}{\Sigma_0} ik\Sigma_a + \frac{2\pi G i \Sigma_a k}{|k|} + 2\Omega v_{ay}.$$

$$-i\omega\, v_{ay} = -(2-q)\,\Omega\, v_{ax}$$

This is a linear system of 3 variables and 3 equations:

$$-i\omega \begin{bmatrix} \Sigma_a \\ v_{ax} \\ v_{ay} \end{bmatrix} = \begin{bmatrix} 0 & -ik\Sigma_0 & 0 \\ -\frac{c_s^2}{\Sigma_0} ik + \frac{2\pi G ik}{|k|} & 0 & 2\Omega v_{ay} \\ 0 & -(2-q) & 0 \end{bmatrix} \begin{bmatrix} \Sigma_a \\ v_{ax} \\ v_{ay} \end{bmatrix}$$

which yields the following equation for the oscilla-tion frequency ω:

$$\boxed{\omega^2 = c_s^2 k^2 - 2\pi G \Sigma_0 |k| + 2(2-q)\Omega^2}$$

The first term is due to pressure.

The second term is due to self-gravity.

The third term is due to rotation.

(Notice that in the absence of pressure and self-gravity, the solution is

$$\omega^2 = \kappa^2$$

with $\kappa^2 = 2(2-q)\Omega^2$ being a natural frequency of the system. If a particle or gas parcel is disturbed from its orbit, it will oscillate around its equilibrium orbital position at this frequency. It this thus called epicyclic frequency.)

The system will be:

Stable if $\omega^2 \geq 0$ (ω real → oscillations)
Unstable if $\omega^2 < 0$ (ω imag → part grows exponentially)

Rotational term κ^2 stabilizing in all scales

Pressure term $c_s^2 k^2$ strong stabilization at high k

Self-gravity $-2\pi G \Sigma_0 |k|$ DESTABILIZING

$$k_{crit} = \frac{\pi G \Sigma_0}{c_s^2} \quad \text{for } \omega^2 < 0$$

$\omega^2(k_{crit}) = 0$, yields

$$\frac{c_s \Omega}{G \Sigma_0} = \pi$$

Toomre Q

$$Q = \frac{c_s \Omega}{\pi G \Sigma}$$

For instability, $Q < 1$

For typical disk parameters

for $c_s \sim 0.5$ km/s, $Q = 1$ means

$$\Sigma \simeq 1.5 \times 10^3 \; g/cm^2$$

The critical wavelength $\lambda_{crit} = \frac{2c_s^2}{G\Sigma}$

will yield a collapsing mass:

$$M_p \sim \pi \lambda^2 \Sigma \sim \frac{4\pi c_s^4}{G^2 \Sigma} \sim 5 \, M_{Jup}$$

This shows that this mechanism, if operative, should produce gas giant planets.

Show HR 8799 as possible case of GI planets.