Class 11

Lane-Emden Equation

Chandrasekhar Limit

Lane - Emden Equation

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -GM_r \Rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr}$$

$$= -G\left(4\pi r^2 \rho\right)$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho}$$

Poisson Equation

$$\nabla^2\Phi = 4\pi G\rho \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_g}{dr}\right) = 4\pi G\rho$$

$$\Phi_g = U_g/m$$

Now use $P = K\rho^\gamma$

$$\frac{\gamma K}{r^2}\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

write $\rho = \rho_c [D_n(r)]^n \quad 0 < D_N < 1$

$$\left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dD_N}{dr}\right] = -D_N^N$$

$$\underbrace{\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)}_{\text{unit of distance}^2}$$

$$\lambda_n \equiv \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

Define $\quad r \equiv \lambda_n \xi$

So LANE-ENDEM EQUATION

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{dD_N}{d\xi}\right] = -D_N^N}$$

solve for $D_N(\xi)$, you have ρ, and via the politropic eq of state, also P. If the ideal gas law holds, you also have T.

The sun is a polytrope of index $n=3$

Chandrasekhar limit

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad ; \quad \frac{dP}{dr} = -\frac{G M_r}{r^2}\rho$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -G M_r \Rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G \cdot 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Eq of state for degenerate matter:

$P = K_1 \rho^{5/3}$ (non-relativistic) $n = 3/2$

$P = K_2 \rho^{4/3}$ (relativistic) $n = 3$

$P = K\rho^{\gamma} = K\rho^{(n+1/n)}$ Polytropic relation

$n = 3/2$ for non-relativistic
$n = 3$ for relativistic

write density as $\rho = \rho_c \theta^n$

ρ_c : core density
$\theta \equiv 1$ at center

$$\rho = \rho_c \theta^n$$

$$P = K \rho_c^{\frac{n+1}{n}} \theta^{n+1}$$

$$\frac{1}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho_c \theta^n} \frac{d}{dr} K \rho_c^{\frac{n+1}{n}} \theta^{n+1} \right)$$

$$= \rho_c \theta^n \Rightarrow \frac{1}{4\pi G r^2} \rho_c^{\left(\frac{n+1}{n} - 2\right)} \frac{d}{dr} \left(\frac{r^2}{\theta^n} \frac{d}{dr} \theta^{n+1} \right) = -\theta^n$$

$$\frac{d\theta^{n+1}}{d\xi} = n+1\, \theta^n \frac{d\theta}{d\xi}$$

$\therefore$

$$\frac{n+1}{4\pi G r^2} \rho_c^{\left(\frac{n+1}{n} - 2\right)} \frac{d}{dr} \left(\frac{r^2 \cancel{\theta^n}}{\cancel{\theta^n}} \frac{d}{dr} \theta^{n+1} \right) = -\theta^n$$

$$r = a \xi \qquad a = \left[\frac{(n+1) K \rho_c^{(1-n)/n}}{4\pi G} \right]^{1/2}$$

(has dimension of length)

<u>Lane-Emden equation</u>:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

If θ falls to zero for finite ξ, that's the surface of the star. Call this ξ_1

$$R = a\,\xi_1$$

$$R \propto \rho_c^{\frac{1-n}{2n}}$$

$$M = \int_0^R 4\pi r^2 \rho\, dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi$$

Dep of a on ρ_c

$$M \propto \left(\rho_c^{\frac{1-n}{n}}\right)^3 \rho_c$$

$$M \propto \rho_c^{(3-n)/2n}$$

Put $n = 3/2$ for non-relativistic

$$R \propto \rho_c^{-1/6} \quad ; \quad M \propto \rho_c^{1/2}$$

$$\boxed{R_c \propto M^{-1/3}}$$

Mass-radius relationship of degenerate matter.

Relativistic: $n = 3$

$M \propto \rho_c^{(3-n)/2n}$: Mass independant of density.

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n \xi^2 \qquad \left(\int_0^{\xi_1}\right)$$

$$\left(\xi^2 \frac{d\theta}{d\xi}\right)\Bigg|_{\xi=\xi_1} - \cancel{\left(\xi^2 \frac{d\theta}{d\xi}\right)\Bigg|_{\xi=0}} = -\int_0^{\xi_1} \theta^n \xi^2 \, d\xi$$

$$\int_0^{\xi_1} \xi^2 \theta^n \, d\xi = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \qquad -\frac{2}{3}$$

So. $$M = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n \, d\xi$$

$$= 4\pi a^3 \rho_c \, \xi_1^2 \, |\theta'(\xi_1)|$$

$$a = \left[\frac{(n+1) K \rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2} \qquad K_2 = \frac{3^{1/3}}{8\pi^{1/3}} \frac{hc}{m_H^{4/3} \mu_e^{4/3}}$$

$$M = 4\pi\left(\frac{(n+1)\rho_c^{(1-n)/n}}{4\pi G}\right)^{3/2} K_2^{3/2} \rho_c \xi_1^2 |\theta'(\xi_1)|$$

$n=3$

$$= 4\pi\left(\frac{\cancel{\rho_c^{-2/3}}}{\pi G}\right)^{3/2} \cdot K_2^{3/2} \cancel{\rho_c} \xi_1^2 |\theta'(\xi_1)|$$

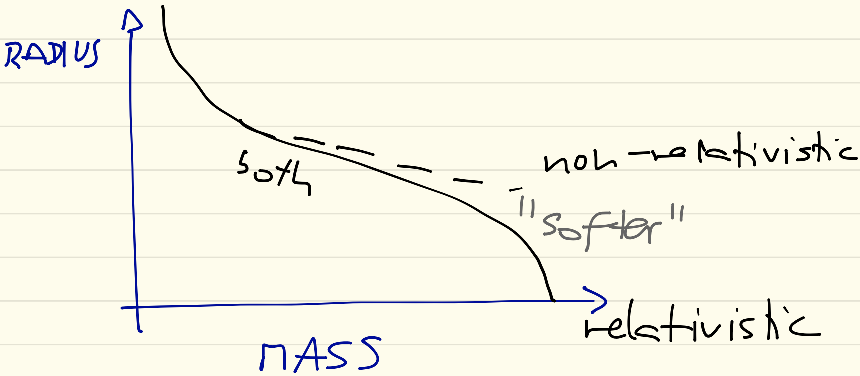
$$= 4\pi\left(\frac{1}{\pi G}\right)^{3/2} \xi_1^2 |\theta'(\xi_1)| \cdot \frac{3^{1/2}}{\pi^{1/2}} \cdot \left(\frac{hc}{8}\right)^{3/2} m_H^{-2} \mu_e^{-2}$$

$$M = \frac{\sqrt{6}}{32\pi}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{2}{\mu_e}\right)^2 \frac{\xi_1^2 |\theta'(\xi_1)|}{m_H^2}$$

$\xi_1^2 |\theta'(\xi_1)| \Rightarrow$ solve numerically: 2.018

$$\boxed{M \simeq 1.46\left(\frac{2}{\mu_e}\right)^2 M_\odot}$$

Chandrasekhar mass-limit



Limiting mass for which the radius goes to zero. (But wouldn't $\xi_1 = 0$ then?

Relativistic when $\rho \simeq 10^6 \frac{g}{cm^3}$

A ton per cm^3... Take that as typical.
So, for mass $10^{33} g$, $R \sim 10^4$ km.