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Physics of Circumstellar Disks

Disks are objects in steady state. They are gaseous objects, so we need to solve the equations of hydrodynamics in a central potential.

$$\frac{\partial \rho}{\partial t} = -(u\cdot\nabla)\rho - \rho\nabla\cdot u$$

$$\frac{\partial u}{\partial t} = -(u\cdot\nabla)u - \frac{1}{\rho}\nabla p - \nabla\phi$$

$$p = \rho c_s^2/\gamma$$

$$\phi = -\frac{GM_*}{r}$$

Let us find the steady-state solution.

$$\frac{\partial \rho}{\partial t} = -u_R\frac{\partial}{\partial R}\rho - \frac{u_\phi}{R}\frac{\partial}{\partial\phi}\rho - u_z\frac{\partial}{\partial z}\rho - \rho\left[\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}\right]$$

$$\frac{\partial u_R}{\partial t} = -u_R\frac{\partial u_R}{\partial R} - \frac{u_\phi}{R}\frac{\partial u_R}{\partial\phi} - u_z\frac{\partial}{\partial z}u_R + \frac{u_\phi^2}{R} - \frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{GM}{r^3}R$$

$$\frac{\partial u_\phi}{\partial t} = -u_R\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\frac{\partial}{\partial\phi}u_\phi - u_z\frac{\partial}{\partial z}u_\phi - \frac{u_\phi u_R}{R} - \frac{1}{\rho R}\frac{\partial p}{\partial\phi}$$

$$\frac{\partial u_z}{\partial t} = -u_R\frac{\partial u_z}{\partial R} - \frac{u_\phi}{R}\frac{\partial}{\partial\phi}u_z - u_z\frac{\partial}{\partial z}u_z - \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GM}{r^3}z$$

For steady state, set $\boxed{\frac{\partial}{\partial t} = 0}$.

Also, azimuthal symmetry. It's a disk. $\partial/\partial\phi = 0$

Consider $\begin{cases} \text{vertical equilibrium} & u_z = 0 \\ \text{centrifugal balance} & u_r = 0 \end{cases}$

$$\cancel{\frac{\partial \rho}{\partial t}} = -\cancel{u_R \frac{\partial}{\partial R}\rho} - \cancel{\frac{u_\phi}{R}\frac{\partial}{\partial \phi}\rho} - \cancel{u_z \frac{\partial}{\partial z}\rho} - \rho\left[\cancel{\frac{\partial u_R}{\partial R}} + \cancel{\frac{u_R}{R}} + \cancel{\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}} + \cancel{\frac{\partial u_z}{\partial z}}\right]$$

$$\cancel{\frac{\partial u_R}{\partial t}} = -\cancel{u_R \frac{\partial u_R}{\partial R}} - \cancel{\frac{u_\phi}{R}\frac{\partial u_R}{\partial \phi}} - \cancel{u_z \frac{\partial u_R}{\partial z}} + \frac{u_\phi^2}{R} - \frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{GM}{r^3}R$$

$$\cancel{\frac{\partial u_\phi}{\partial t}} = -\cancel{u_R \frac{\partial u_\phi}{\partial R}} - \cancel{\frac{u_\phi}{R}\frac{\partial u_\phi}{\partial \phi}} - \cancel{u_z \frac{\partial u_\phi}{\partial z}} - \cancel{\frac{u_\phi u_R}{R}} - \cancel{\frac{1}{\rho R}\frac{\partial p}{\partial \phi}}$$

$$\cancel{\frac{\partial u_z}{\partial t}} = -\cancel{u_R \frac{\partial u_z}{\partial R}} - \cancel{\frac{u_\phi}{R}\frac{\partial u_z}{\partial \phi}} - \cancel{u_z \frac{\partial u_z}{\partial z}} - \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GM}{r^3}z$$

We are left with quite little, considering only u_ϕ:

$$\frac{u_\phi^2}{R} = \frac{GM}{r^3}R + \frac{1}{\rho}\frac{\partial p}{\partial R} \qquad \text{(Radial)}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{GM}{r^3}z \qquad \text{(vertical)}$$

The first equation gives the condition of centrifugal balance. Rewrite it to

$$\frac{GM}{r^3}R = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{u_\phi^2}{R}$$

$$\frac{GM}{r^3}z = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

To highlight why disks are flat. It is the influence of the centrifugal force. Gravity makes things spherical. Rotation makes cylindrical.

A pressure-supported object is spherical
A rotation (centrifugal) supported object is flat

The vertical equation gives the condition of vertical balance

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{GM}{r^3} z$$

$p = \rho c_s^2$ assume isothermal or simply $c_s = c_s(R)$ not z

$$c_s^2 \frac{\partial \ln \rho}{\partial z} = -\frac{GM}{r^3} z \quad \therefore \quad \frac{\partial \ln \rho}{\partial z} = -\frac{GM}{c_s^2}\frac{z}{r^3}$$

$$\ln \rho = -\frac{GM}{c_s^2}\int \frac{z\, dz}{(R^2+z^2)^{3/2}} \quad \therefore \quad \rho(R,z) = \rho(R) \exp\left(\frac{GM}{c_s^2}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{R}\right]\right)$$

Write

$$\sqrt{R^2+z^2} = R\sqrt{1+(z/R)^2}$$

And Taylor-expand the root: $\frac{1}{\sqrt{1+(z/R)^2}} \approx 1 - \frac{1}{2}\left(\frac{z}{R}\right)^2$

$$\rho(R,z) \simeq \rho(R) \exp\left(\frac{GM}{c_s^2 R}\left(\cancel{1} - \frac{1}{2}\left(\frac{z}{R}\right)^2 - \cancel{1}\right)\right)$$

$$\rho(R,z) \simeq \rho(R) \exp\left(-\frac{GM}{2c_s^2 R^3} z^2\right)$$

This quantity $\frac{c_s^2 R^3}{GM}$ has dimension of L^2, and thus defines a scale height

$$\frac{c_s^2 R^3}{GM} = H^2 \quad \therefore \quad H = \frac{c_s}{\sqrt{\frac{GM}{R^3}}}$$

So $\sqrt{\frac{GM}{R^3}}$ must have unit of frequency. It is the Keplerian angular frequency

$$\Omega_k \equiv \sqrt{\frac{GM}{R^3}}$$

So $\boxed{H \equiv \frac{c_s}{\Omega}}$ is the disk scale height,

$$\boxed{\rho(R,z) = \rho(R)\, e^{-z^2/2H^2}}$$

Notice that $c = \Omega H$ and $v = \Omega r$, so

$$\frac{H}{R} = \frac{c}{\Omega} \cdot \frac{\Omega}{v} = \frac{c}{v} = \frac{1}{Ma}$$

At the position of Jupiter, $T \approx 180K$ and $v \simeq 10$ Km/s

$c \simeq 500$ m/s, and thus $\frac{H}{R} \approx 0.05$ <u>The disk is thin</u>.

This quantity is usually called $h = H/R$ disk aspect ratio

The first equation gives the condition of centrifugal balance

$$\frac{u_\phi^2}{R} = \frac{GM}{r^3}R + \frac{1}{\rho}\frac{\partial p}{\partial R}$$

With

$$\rho(R,z) = \rho_0\left(\frac{R}{R_0}\right)^{-p} \exp\left(\frac{GM}{c_s^2}\left[\frac{1}{\sqrt{R+z^2}} - \frac{1}{R}\right]\right)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial R} = \left[\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} + \frac{\partial c_s^2}{\partial R}\right] = c_s^2\left[\frac{\partial \ln \rho}{\partial r} + \frac{\partial \ln c_s^2}{\partial r}\right]$$

$$\ln\rho = \ln\rho_0 - p\ln R + p\ln R_0 + \frac{GM}{c_s^2}\frac{1}{\sqrt{R^2+z^2}} - \frac{GM}{c_s^2 R}$$

$$\frac{\partial \ln\rho}{\partial R} = -\frac{p}{R} + \frac{GM}{c_s^2R^2} - \frac{1}{\cancel{2}r^3}\frac{GM}{c_s^2}\cdot\cancel{2}R - \left(\frac{GM}{\sqrt{R^2+z^2}} - \frac{GM}{R}\right)\frac{1}{c_s^2}\frac{\partial \ln c_s^2}{\partial r}$$

$$\frac{\partial \ln c_s^2}{\partial R} = \frac{R}{R}\frac{\partial \ln c_s^2}{\partial R} = \frac{1}{R}\frac{\partial \ln c_s^2}{\partial \ln R} = -\frac{q}{R}$$

$$\therefore \frac{1}{\rho}\frac{\partial p}{\partial R} = -\frac{c_s^2}{R}(p+q) + \frac{GM}{R^2} - \frac{GM}{r^3}R - \frac{GM}{2}\frac{z^2}{R^3}\frac{q}{R}$$

So,

$$\frac{u_\phi^2}{R} = \frac{GM}{r^3}R + \frac{1}{\rho}\frac{\partial p}{\partial R}$$

$$\frac{u_\phi^2}{R} = \frac{GM}{R^2} - \frac{c_s^2}{R}(p+q) - \frac{GM}{2R^3} z^2 \frac{q}{R^2}$$

By definition $u_\phi \equiv \Omega R$

$$\Omega^2 = \frac{GM}{R^3} - \frac{c_s^2}{R^2}(p+q) - \frac{GM}{2\,2^3} z^2 \frac{q}{R^2}$$

$$\Omega^2 = \Omega_K^2 - \frac{c_s^2}{R^2}(p+q) - \Omega_K^2 \frac{q}{2}\left(\frac{z}{R}\right)^2$$

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$$\Omega^2 = \Omega_K^2 - \frac{\Omega_K^2 H^2}{R^2}(p+q) - \Omega_K^2 \frac{q}{2}\left(\frac{z}{R}\right)^2$$

$$\Omega^2 = \Omega_K^2\left[1 - \left(\frac{H}{R}\right)^2(p+q) - \frac{q}{2}\left(\frac{z}{R}\right)^2\right]$$

$$\Omega^2 = \Omega_K^2\left[1 - \left(\frac{H}{R}\right)^2\left[p+q + \frac{\cancel{R^2}}{H^2}\frac{q}{2}\frac{z^2}{\cancel{R^2}}\right]\right]$$

$$\Omega^2 = \Omega_K^2\left\{1 - \left(\frac{H}{R}\right)^2\left[p+q + \frac{q}{2}\left(\frac{z}{H}\right)^2\right]\right\}$$

$$\boxed{\Omega = \Omega_K\left\{1 - \frac{1}{2}\left(\frac{H}{R}\right)^2\left[p+q + \frac{q}{2}\left(\frac{z}{H}\right)^2\right]\right\}}$$

if $p = q = 0$, $\Omega^2 = \Omega_K^2 = \frac{GM}{R^3}$

Because $\left(\frac{H}{R}\right)$ is small, the deviations from Keplerian are small.

Notice also that $c = \Omega H$

$$h = \frac{H}{R} = \frac{c}{\Omega R} \therefore \; c_0 \left(\frac{r}{r_0}\right)^{-q/2} \left(\Omega_0 \left(\frac{r}{r_0}\right)^{-3/2}\right)^{-1} \left(\frac{r}{r_0}\right)^{-1}$$

$$h = \frac{c_0 \Omega_0}{r_0} \left(\frac{r}{r_0}\right)^{-\frac{q}{2} + \frac{3}{2} - \frac{2}{2}} \qquad h \propto r^{(1-q)/2}$$

$$\boxed{\begin{array}{l}\text{if } T \propto 1/r \text{ the disk has constant aspect ratio} \\ \text{if } T \text{ falls faster than } 1/r,\ h \text{ falls with distance} \\ \text{if } T \text{ falls slower than } 1/r,\ h \text{ increases with distance}\end{array}}$$

So which one is it? To answer that, calculate disk temperature. Imagine a flat thin disk that absorbs all incoming light and re-emits as blackbody. The flux is

$$F = \int I \sin\theta \cos\phi \, d\Omega$$

Integrate from top half of star only $\quad -\pi/2 < \phi < \pi/2$

$0 < \theta < \sin^{-1}(R_*/r)$

$$\text{So } F = I_* \int_{-\pi/2}^{\pi/2} \cos\phi \, d\phi \int_0^{\sin^{-1}(R_*/r)} \sin^2\theta \, d\theta$$

Solve integral

$$F = I_* \left[\sin^{-1}\left(\frac{R_*}{r}\right) - \left(\frac{R_*}{r}\right) \sqrt{1 - \left(\frac{R_*}{r}\right)^2} \right]$$

Taylor expand $\frac{R_*}{r} \ll 1$ (far from star) $\quad \mathrm{asin}(x) \approx x + \frac{x^3}{6}$

$$F = I_* \left[R_*/r - \frac{R_*}{r}\left(1 - \frac{1}{2}\frac{R_*^2}{r^2}\right) \right] = I_* \left[\frac{R_*^3}{2r^3} \right]$$

$$F = I_* \left[R_*/r - \frac{R_*}{r}\left(1 - \frac{1}{2}\frac{R_*^2}{r^2}\right)\right] = I_* \left[\frac{R_*^3}{2r^3} \right]$$

$$F = \frac{\sigma T_{disk}^4}{2} \qquad I = \frac{\sigma T_*^4}{\pi}$$

$$\therefore \quad T_d^4 = T_*^4 \; \frac{R_*^3}{\pi} \cdot \frac{1}{r^3}$$

$\boxed{T_d \propto r^{-3/4}}$ for a passively irradiated disk

The disk flares slightly as $h \propto r^{1/8}$

Disk viscosity

Ok, we got the steady state solution, without radial velocities. Such a disk will just rotate and rotate without doing much. Show plot of time dissipation.

The disks evolve in time! This is obvious at first, since the disk has to accrete to complete the formation of the star.

It turns out angular momentum is a conserved quantity

$$\ell = R\, v_\phi = R^2 \Omega = \sqrt{G M_* R}$$

A gas parcel cannot simply "lose" angular momentum. It needs to be transported.

Estimates. At 5 AU vs at surface of Sun

$$\frac{L_{5AU}}{L_{R_\odot}} = \frac{\cancel{\sqrt{GM}}}{\cancel{\sqrt{GM}}} \cdot \sqrt{\frac{a_J}{R_\odot}} \approx 30 \qquad a_J \approx 1000\, R_\odot$$

The disk has about 30x more angular momentum than the star can accomodate, already at 5AU. The mass must lose angular momentum somehow.

1- Viscosity
2- Winds

Viscosity is something for engineers, but in accretion disk theory it occupies the central spot.

Ring A moves faster than B. Friction between them will slow down A and speed up B. Angular momentum is transfered from A to B.

$$\ell = \sqrt{GMr} \qquad L \equiv r$$

lose angular momentum, moves inword
gains angular momentum, moves outward

Let's understand this process. The viscous force is

$$\frac{Du}{Dt} = -\nabla\phi - \frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla\cdot(2\nu\rho S) \qquad S = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\frac{Du}{Dt} = -\nabla\phi - \frac{1}{\rho}\bar{\nabla}p - \frac{1}{\rho}\bar{\nabla}\cdot(2\nu\rho S)$$

$$r\dot{\phi} = -\nu\left(r\frac{\partial^2\dot{\phi}}{\partial r^2} + 3\frac{\partial\dot{\phi}}{\partial r}\right)$$

This term will immediately leed to a radial velocity

The conservation equations are

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma u_r\right) = 0$$

$$\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r(\Sigma r^2\Omega)u_r\right) = \frac{1}{2\pi r}\int \nabla\cdot T \, r\,d\phi \qquad (\mathrm{II})$$

$$T = 2\pi r \cdot \nu\Sigma r\frac{d\Omega}{dr} \cdot r \qquad r \times F_\nu$$

whole lenght — viscous force per length — arm

Both equations can be combined (substitute v_r by $\partial\Sigma/\partial t$ in II)

$$\frac{\partial\Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{(r^2\Omega)'}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\Omega'\right)\right]$$

For Keplerian disks

$$\boxed{\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\nu\Sigma r^{1/2}\right)\right]}$$

Amazingly enough, for ν = cte this equation has an analytical solution (show)

One can also find the expression for the velocity, given $\frac{\partial \Sigma}{\partial t}$

$$v_r = \frac{-3}{\Sigma r^{1/2}} \frac{\partial}{\partial t} \left(\nu \Sigma r^{1/2} \right)$$

for constant Σ and ν, that yields

$$\boxed{u_r = -\frac{3\nu}{2r}}$$

And the mass accretion rate

$$\dot{m} = \oint\!\!\oint \Sigma u \cdot \hat{n} \, dA = -2\pi r \Sigma u_r$$

$$\boxed{\dot{m} = 3\pi \nu \Sigma}$$

One can also define the viscous timescale.

$$[\nu] = \frac{cm^2}{s} \quad \therefore \quad \boxed{\tau_\nu = \frac{R^2}{\nu}}$$

These equations allows for comparisons to observations. Observations of T-Tauri reveal mass accretion rates of

$$\dot{m} \approx 10^{-8} \, M_\odot/yr$$

Using this and typical MMSN values ($\Sigma \approx 10^3$ g/cm^2)

$$\nu \approx \frac{\dot{M}}{10\Sigma} \approx 10^{14} \frac{\text{cm}^2}{\text{s}}$$

Molecular viscosity:

$$\nu \equiv \lambda \cdot v_{TH}$$

$$\lambda = \frac{1}{n \cdot \sigma_{coll}} \qquad \left.\begin{array}{l} n \approx 10^{14}\ \text{cm}^{-3} \\ \sigma_{coll} \approx 2 \times 10^{-15}\ \text{cm}^{-2} \end{array}\right\} \lambda \approx 10\ \text{cm}$$

$v_{TH} \approx c_s \approx 1$ km/s

$\therefore\ \nu \approx 10^7\ \text{cm s}^{-1}$

Seven orders of magnitude lower than required. Forming a star would take

$$\tau \simeq \frac{R^2}{\nu} = \frac{(10^{14})^2}{10^7} \approx 10^{21}\,\text{s} \approx 10^{13}\ \text{yrs}.$$

Much longer than a Hubble time

Another mechanism, that acts as an effective viscosity must be involved.

The Shakura-Sunyaev model

Turbulence is a source of viscosity

$$\rho \partial_t u_i = \rho F_i - \partial_j (p \delta_{ij} + \rho u_i u_j)$$

For mean quantities

Decompose in mean and fluctuation

$$\rho \partial_t (\bar{u}_i + u_i') = \rho F_i - \partial_j (\bar{p} \delta_{ij} + p' \delta_{ij} + \rho (\bar{u}_i + u_i')(\bar{u}_j + u_j'))$$

Average out $\langle u_i' \rangle = 0$

$$\rho \partial_t \bar{u}_i = \rho F_i - \partial_j (\bar{p} \delta_{ij} + \rho \bar{u}_i \bar{u}_j + \rho \langle u_i u_j \rangle) = 0$$

This term behaves like a viscous tensor, and transports angular momentum.

Effectively, it's a diffusion of momentum, coming from the advection term, advecting momentum out of a gaussian surface.

Reynolds stress behave like a viscous term

Subs. continuity equation on ang. momentum eq. and set $\frac{\partial}{\partial \varphi} = 0$

$$\frac{\partial L}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R L u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu L \frac{d\ln\Omega}{d\ln R}\right)$$

$$\frac{\partial \overline{L_\phi}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \overline{L_\phi}\, \bar{u}_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \rho \langle u'_\phi\, u'_r\rangle\right)$$

$$r^2 \rho \langle u'_r\, u'_\phi\rangle = \nu L \frac{d\ln\Omega}{d\ln R}$$

$$\nu = -\left(\frac{d\ln\Omega}{d\ln s}\right)^{-1} \frac{R^{\phi r}}{\rho\Omega} \quad ; \quad R^{\phi r} = \rho \langle u'_r u'_\phi\rangle$$

Equivalence turbulence and viscosity. This "momentum diffusion" is in fact a pressure. So, we can write

$$\boxed{R^{\phi r} = \alpha P}$$

The stress is proportional to the pressure. The causality is correct here. The stress generates a pressure

$$\nu = -\left(\frac{d\ln\Omega}{d\ln r}\right)\frac{\alpha P}{\rho\Omega} = \frac{2}{3}\alpha \frac{\cancel{\rho} c_s^2}{\cancel{\rho}\Omega}$$

$$\boxed{\nu \simeq \alpha c_s H}$$

Two different ways to write it. Both physical. The viscosity is prop to a lenght. H is the sonic scale, the last isotropic length. The size of the eddies is H. The speeds are still thermal, so

$$[\nu] = LU \Rightarrow \nu = \alpha c_s H$$

If α is constant, it's possible to solve analytically for several disk phenomena.

Time scales: $\tau = \frac{r^2}{\nu} = \left(\frac{H}{R}\right)^{-2} \frac{1}{\alpha \Omega}$

For $\left(\frac{H}{R}\right) \simeq 0.05$ $\alpha \approx 10^{-2}$

$$\dot{m} = 3\pi \nu \Sigma = 3\pi \alpha c_s H \Sigma \quad \therefore \alpha \approx 10^{-2}$$

Turbulence

What leads to turbulence – Rayleigh criterion

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + g$$

Decompose the flow into base (time-indepent) and perturbation $u(\vec{x},t) = \bar{u}(\vec{x}) + u'(\vec{x},t)$, with $\bar{u}_r = 0$ and $\bar{u}_\phi = \Omega r \gg u'_\phi$

$$\frac{\partial u_r}{\partial t} + u_\phi \frac{\partial}{\partial \phi} u_r - \frac{u_\phi^2}{r} = \Omega^2 r \xrightarrow{u = \bar{u} + u'} \frac{\partial u'_r}{\partial t} - \cancel{\Omega^2 r} - 2\Omega u'_\phi = -\cancel{\Omega^2 r}$$

$$\boxed{\frac{\partial u_r'}{\partial t} - 2\Omega u_\phi = 0}$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial}{\partial r} u_\phi + \frac{u_\phi u_r}{r} = 0 \qquad \frac{\partial u_\phi'}{\partial t} + u_r' \frac{\partial}{\partial r}(\Omega r) + \frac{\Omega r u_r}{r}$$

$$\frac{\partial u_\phi'}{\partial t} + u_r'\left[\Omega + \frac{r}{\Omega}\frac{\partial \Omega}{\partial r}\right] + \Omega u_r \Rightarrow \boxed{\frac{\partial u_\phi'}{\partial t} + (2-q)\,\Omega u_r = 0}$$

$$q \equiv -\frac{d\ln\Omega}{d\ln r}$$

$$\Omega = \Omega_0\left(\frac{r_0}{r}\right)^q$$

The perturbation equations are thus:

$$\partial_t u_r' - 2\Omega u_\phi' = 0 \quad \text{and} \quad \partial_t u_\phi' + \Omega(2-q)\, u_r' = 0$$

Now consider the perturbation as a Fourier mode $\psi' = \hat{\psi}_0 e^{-i(\omega t - Kx)}$

$$-i\omega\, \hat{u}_r - 2\Omega\, \hat{u}_\phi = 0 \qquad \rightarrow \quad \hat{u}_\phi = -\frac{i\omega}{2\Omega}\hat{u}_r$$

$$-i\omega\, \hat{u}_\phi + \Omega(2-q)\, \hat{u}_r = 0$$

$$-i\omega\left(\frac{i\omega}{2\Omega}\right)\hat{u}_r + \Omega(2-q)\, \hat{u}_r = 0$$

$$-\frac{\omega^2}{2\Omega}\cancel{\hat{u}_r} + \Omega(2-q)\cancel{\hat{u}_r} = 0$$

$$\omega^2 - 2(2-q)\,\Omega^2 = 0$$

$$\boxed{\omega^2 = 2\Omega^2(2-q)}$$ if $q<2$; then the flow is stable.

This frequency is called epicyclic frequency, $\kappa = \Omega\sqrt{2(2-q)}$

$q = -\frac{d\ln\Omega}{d\ln r} < 2$ is the case for keplerian Disks. So, they are stable.

$$\boxed{\left|\frac{d\ln\Omega}{d\ln r}\right| < 2 \qquad \text{Stability}}$$

Equivalent statement: the angular momentum must increase outward

$$L = r^2\Omega$$

$$\frac{d\,\Omega r^2}{dr} = r^2\frac{d\Omega}{dr} + 2\Omega r$$

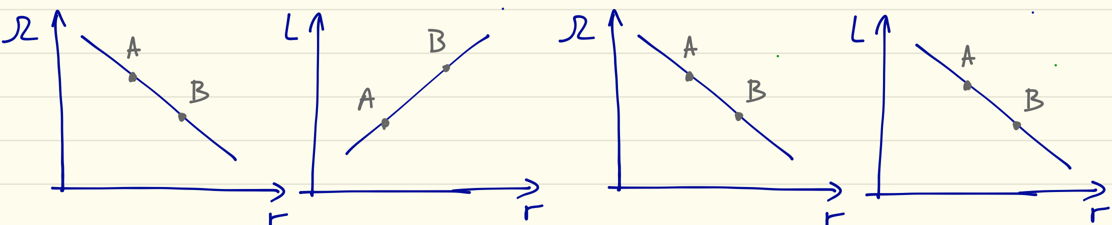
$$= r^2\frac{\Omega}{\Omega}\frac{r}{r}\frac{d\Omega}{dr} + 2\Omega r$$

$$= r^{\cancel{2}}\frac{\Omega}{\cancel{r}}\frac{d\ln\Omega}{d\ln r} + 2\Omega r$$

$$= \Omega r\left(2 + \frac{d\ln\Omega}{d\ln r}\right) = \Omega r\,(2-q)$$

The sign of $\frac{dL}{dr}$ is the same sign of $(2-q)$. So, the condition $2-q>0$ for stability is equivalent to $dL/dr > 0$, or L increasing with distance. Let us understand it (show slides)

STABILITY | INSTABILITY



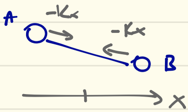
friction between A and B makes A slow down. It loses (angular) momentum and jumps to an orbit of lower L, inwards.

friction between A and B makes A slow down. It loses (angular) momentum and jumps to an orbit of lower L, **outwards**.

Since the orbit of lower angular momentum is outward, if A loses angular it must move outward. B gaining angular momentum must move inward. The rings swap and the situation is unstable.

Magnetorotational Instability

Consider now that the gas parcels are connected by a tether with a restoring force $-Kx$



$$\frac{\partial u_x}{\partial t} - 2\Omega u_y = -Kx$$

$$\frac{\partial u_y}{\partial t} + (2-q)\Omega u_x = -Ky$$

$$\ddot{x} - 2\Omega \dot{y} + Kx = 0$$
$$\ddot{y} + (2-q)\Omega \dot{x} + Ky = 0$$

Consider the Lagrangian displacement $x(t) = \bar{x} + \xi(t)$, with $\bar{x} = 0$ and $\xi = x_0 e^{i\omega t}$

$$\dot{x} = i\omega \xi \qquad \ddot{x} = -\omega^2 \xi$$

$$-\xi_x \omega^2 - 2\Omega \xi_y i\omega + K\xi_x = 0$$
$$-\xi_y \omega^2 + (2-q)\Omega \xi_x i\omega + K\xi_y = 0$$

$$\boxed{\omega^4 - (2K + \kappa^2)\omega^2 + K(K - 2q\Omega^2) = 0}$$

Bi-quadratic equation. Solution: $\omega^2 = \frac{(2K + \kappa^2)^2 \pm \sqrt{4K(K - 2q\Omega^2)}}{2}$

Condition for instability

$$\boxed{K - 2q\Omega^2 < 0}$$

If $K \Rightarrow 0$, the it is simply $q > 0$, i.e., the angular velocity decreasing outward. That is satisfied in Keplerian disks.

Magnetic field $\equiv$ spring

$$\frac{\partial B'}{\partial t} = -(u \cdot \nabla) B + (B \cdot \nabla) u - \cancel{B(\nabla \cdot u)}$$

$$\delta B = B\, iK\xi$$

$$\frac{(B \cdot \nabla) B}{\mu_0 \rho} \qquad \frac{\bar{B}\, \partial B}{\mu_0 \rho} = \frac{B\, iK\, B'\xi}{\mu_0 \rho} = -\frac{K^2 B^2}{\mu_0 \rho}\xi = -(K \cdot v_A)^2 \xi$$

Equivalence between MRI and springs. The magnetic tension behaves EXACTLY like a restoring force, provided $K \equiv (k \cdot v_A)^2$. So, the instability condition is

$$\boxed{(k \cdot v_A)^2 - 2q\Omega^2 < 0}$$

Discuss weak and strong field limit.

Thermal instabilities

VSI $\Rightarrow$ Vertical violation of Rayleigh (Ang mom change)
COV $\Rightarrow$ Epicyclic by buoyancy
ZVI $\Rightarrow$ Resonance epicyclic and buoyancy.

k_z only, $\tau = \infty$

$$\frac{\partial \rho}{\partial t} + (u \cdot \nabla)\rho = -\rho \nabla \cdot u$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho}\nabla p + g$$

$$\frac{\partial p}{\partial t} + (u \cdot \nabla) p = -\gamma p \nabla \cdot u$$

$$M \cdot \Lambda = 0$$

$$\Lambda = \begin{bmatrix} \rho' \\ u_r' \\ u_\phi' \\ u_z' \\ p' \end{bmatrix}$$

pert : $u_r = u_r'$; $u_\phi = \Omega r + u_\phi'$
$p = p_0 + p'$; $\rho = \rho_0 + \rho'$

$$-i\omega \rho' + u_r' \partial_r \rho_0 + \rho_0 i k_z u_z' = 0$$

$$-i\omega u_r' - 2\Omega u_\phi' - \frac{\rho'}{\rho_0^2}\partial_r p_0 = 0$$

$$-i\omega u_\phi' + \Omega(2-q) u_r' = 0$$

$$-i\omega u_z' + \frac{ik}{\rho_0} p' = 0$$

$$-i\omega p' + u_r' \partial_r p_0 + \rho_0 c^2 i k_z u_z' = 0$$

$$M = \begin{bmatrix} -i\omega & \rho_0 A/c & 0 & \rho_0 i k_z & 0 \\ -Bc/\rho_0 & -i\omega & -2\Omega & 0 & 0 \\ 0 & \Omega(2-q) & -i\omega & 0 & 0 \\ 0 & 0 & 0 & -i\omega & i k_z/\rho_0 \\ 0 & \rho_0 c B & 0 & \rho_0 c^2 i k_z & 0 \end{bmatrix}$$

$A = c\,\partial_r \ln p$

$B = \gamma^{-1} c\,\partial_r \ln p$

$$\omega^4 - \omega^2 (AB + c^2k^2 + \kappa^2) + c^2k^2(\kappa^2 + AB - B^2) = 0$$

Collect only the $c \to \infty$ infinity terms $\omega^2 \cancel{k^2} = \cancel{k^2}(\kappa^2 + AB - B^2)$

$$\omega^2 = \kappa^2 + AB - B^2$$

$$AB - B^2 = \frac{1}{p}\frac{dp}{dr}\left(\frac{1}{p}\frac{dp}{dr} - \frac{1}{\gamma p}\frac{dp}{dr}\right) = N^2$$

$$\boxed{\omega^2 = \kappa^2 + N^2}$$ N^2: Brunt–Vaisala frequency

with cooling

$$\omega^3 + \frac{i}{\gamma\tau}\omega^2 - \omega(\kappa^2 + N^2) - \frac{i}{\gamma\tau}\kappa^2 = 0$$

Allows for growing root. show results.

Practical example. Code your own 1D finite-difference code.

Solve numerically the advection-diffusion equation

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \nu\frac{\partial^2 f}{\partial x^2}$$

Benchmark against the analytical solution

$$\frac{\partial f}{\partial t} = -f K^2 \qquad \frac{\partial \ln f}{\partial t} = -K^2\nu \qquad f = f_0 e^{-tK^2\nu} = f_0 e^{-t/\tau_\nu}$$

$$f = A\sin(Kx - \omega t)$$

$$\frac{\partial f}{\partial t} = -vAK\cos(Kx - \omega t)$$

$$f = -vAK\int \cos(Kx - \omega t)\,dt$$

$$f = \frac{v}{\omega}AK\sin(Kx - \omega t) =$$

$$\frac{vK}{\omega} = 1 \qquad \omega = vK \qquad \therefore\; f(t) = A\sin(K(x - vt))$$

$$\boxed{f = f_0 e^{-t/\tau_\nu}\sin(-K(vt - x)) \qquad \tau_\nu = \frac{1}{K^2\nu}}$$