Class 8: <u>structure of disks</u> 2/18/20

Disks are objects in steady state. They are gaseous objects, so we need to solve the equations of hydrodynamics in a central potential

$$\frac{\partial \rho}{\partial t} = -(u \nabla)\rho - \rho \nabla u$$

$$\frac{\partial u}{\partial t} = -(u \nabla) u - \frac{1}{\rho}\nabla p - \nabla \phi$$

$$p = \rho c_s^2/\gamma$$

$$\phi = -\frac{GM_*}{r}$$

$$\phi = \phi_* + \phi_{disk}$$

$$\phi_* = -\frac{GM_*}{r}$$

$$\nabla^2 \phi_{disk} = 4\pi G \rho$$

but assume for now star dominates

Let us find the steady-state solution

$$\frac{\partial \rho}{\partial t} = -u_R \frac{\partial}{\partial R}\rho - \frac{u_\phi}{R}\frac{\partial}{\partial \phi}\rho - u_z \frac{\partial}{\partial z}\rho - \rho\left[\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z}\right]$$

$$\frac{\partial u_R}{\partial t} = -u_R \frac{\partial u_R}{\partial R} - \frac{u_\phi}{R}\frac{\partial u_R}{\partial \phi} - u_z \frac{\partial u_R}{\partial z} + \frac{u_\phi^2}{R} - \frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{GM}{r^3}R$$

$$\frac{\partial u_\phi}{\partial t} = -u_R \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\frac{\partial u_\phi}{\partial \phi} - u_z \frac{\partial u_\phi}{\partial z} - \frac{u_\phi u_R}{R} - \frac{1}{\rho R}\frac{\partial p}{\partial \phi}$$

$$\frac{\partial u_z}{\partial t} = -u_R \frac{\partial u_z}{\partial R} - \frac{u_\phi}{R}\frac{\partial u_z}{\partial \phi} - u_z \frac{\partial u_z}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GM}{r^3}z$$

For steady state, set $\boxed{\frac{\partial}{\partial t} = 0}$

Also, azimuthal symmetry It's a disk $\partial/\partial\phi = 0$

Consider $\begin{cases} \text{vertical equilibrium} & u_z = 0 \\ \text{centrifugal balance} & u_r = 0 \end{cases}$

$$\cancel{\frac{\partial \rho}{\partial t}} = -u_R \cancel{\frac{\partial}{\partial R}}\rho - \frac{u_\phi}{R}\cancel{\frac{\partial}{\partial \phi}}\rho - u_z \cancel{\frac{\partial}{\partial z}}\rho - \rho\left[\cancel{\frac{\partial u_R}{\partial R}} + \cancel{\frac{u_R}{R}} + \cancel{\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}} + \cancel{\frac{\partial u_z}{\partial z}}\right]$$

$$\cancel{\frac{\partial u_R}{\partial t}} = -\cancel{u_R\frac{\partial}{\partial R}u_R} - \cancel{\frac{u_\phi}{R}\frac{\partial}{\partial \phi}u_R} - \cancel{u_z\frac{\partial}{\partial z}u_R} + \frac{u_\phi^2}{R} - \frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{GM}{r^3}R$$

$$\cancel{\frac{\partial u_\phi}{\partial t}} = -\cancel{u_R\frac{\partial}{\partial R}u_\phi} - \cancel{\frac{u_\phi}{R}\frac{\partial}{\partial \phi}u_\phi} - \cancel{u_z\frac{\partial}{\partial z}u_\phi} - \cancel{\frac{u_\phi u_R}{R}} - \cancel{\frac{1}{\rho R}\frac{\partial p}{\partial \phi}}$$

$$\cancel{\frac{\partial u_z}{\partial t}} = -\cancel{u_R\frac{\partial}{\partial R}u_z} - \cancel{\frac{u_\phi}{R}\frac{\partial}{\partial \phi}u_z} - \cancel{u_z\frac{\partial}{\partial z}u_z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GM}{r^3}z$$

We are left with quite little, considering only $u_\phi$

$$\frac{u_\phi^2}{R} = \frac{GM}{r^3}R + \frac{1}{\rho}\frac{\partial p}{\partial R} \qquad \text{(Radial)}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{GM}{r^3}z \qquad \text{(vertical)}$$

The first equation gives the condition of centrifugal balance

Rewrite it to

$$\frac{GM}{r^3}R = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{u_\phi^2}{R}$$

$$\frac{GM}{r^3}z = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

To highlight why disks are flat
It is the influence of the centrifugal force Gravity makes things spherical Rotation makes cylindrical

A pressure-supported object is spherical
A rotation (centrifugal)-supported object is flat

The vertical equation gives the condition of vertical balance

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{GM}{r^3}z$$

$p = \rho c_s^2$ assume isothermal or simply $c_s = c_s(R)$ not $z$

$$c_s^2 \frac{\partial \ln \rho}{\partial z} = -\frac{GM}{r^3}z \qquad \frac{\partial \ln \rho}{\partial z} = -\frac{GM}{c_s^2}\frac{z}{r^3}$$

$$\ln \rho = -\frac{GM}{c_s^2}\int \frac{z\,dz}{(R^2+z^2)^{3/2}} \qquad \rho(R,z) = \rho(R)\exp\left(\frac{GM}{c_s^2}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{R}\right]\right)$$

Write
$\sqrt{R^2+z^2} = R\sqrt{1+(z/R)^2}$

And Taylor-expand the root $\dfrac{1}{\sqrt{1+(z/R)^2}} \approx 1 - \dfrac{1}{2}\left(\dfrac{z}{R}\right)^2$

$$\rho(R,z) \simeq \rho(R)\exp\left(\frac{GM}{c_s^2 R}\left(\cancel{1} - \frac{1}{2}\left(\frac{z}{R}\right)^2 \cancel{-1}\right)\right)$$

$$\rho(R,z) \simeq \rho(R)\exp\left(-\frac{GM}{2c_s^2R^3}z^2\right)$$

This quantity $\dfrac{c_s^2R^3}{GM}$ has dimension of $L^2$, and thus defines a scale height

$$\frac{c_s^2 R^3}{GM} = H^2 \quad . \quad H = \frac{c_s}{\sqrt{\frac{GM}{R^3}}}$$

So $\sqrt{\frac{GM}{R^3}}$ must have unit of frequency. It is the Keplerian angular frequency

$$\Omega_k \equiv \sqrt{\frac{GM}{R^3}}$$

So $\boxed{H \equiv \frac{c_s}{\Omega}}$ is the disk scale height,

$$\boxed{\rho(R,z) = \rho(R)\, e^{-z^2/2H^2}}$$

Notice that $c = \Omega H$ and $v = \Omega r$, so

$$\frac{H}{R} = \frac{c}{\Omega}\frac{\Omega}{v} = \frac{c}{v} = \frac{1}{Ma}$$

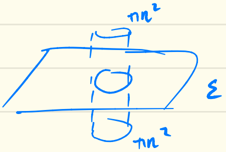
At the position of Jupiter, $T \approx 180K$ and $v \simeq 10$ Km/s

$c \approx 500$ m/s, and thus $\frac{H}{R} \approx 0.05$ The disk is thin

This quantity is usually called $h = H/R$ disk aspect ratio

$$c_s^2 = \frac{kT}{\mu m_p}$$

Check that disk gravity does not impact structure:




$$\nabla\cdot g = -4\pi G\Sigma\,\delta(z)$$

$$\int \nabla\cdot g\, dV = -4\pi G\Sigma \int \delta(z)\, dV$$

$$\cancel{2}\, g_z \cdot \pi \cancel{R^2} = \overset{2}{\cancel{4}}\pi G\Sigma\, \cancel{\pi} R^2 \qquad \boxed{g_z = -2\pi G\Sigma}$$

Compare this with vertical gravity of star at $z \sim h$

$$\frac{GM}{r^3}\cdot h \quad = \quad \frac{2\pi G\Sigma}{\frac{GM}{r^3}\cdot H} = \frac{2\pi r^2\Sigma}{M(H/r)} = \frac{M_{disk}}{M_\star\,(H/R)}\cdot 2$$

$M_{disk} \sim \pi r^2\Sigma$ →

$$\therefore \frac{M_{disk}}{M_\star} \ll \frac{1}{2}\left(\frac{h}{r}\right)$$

for MMSN $M_{disk} \approx 0.01\, M_\star$

∴ satisfied.

But beware of more massive disks.

Radial:

$$\frac{u_\phi^2}{R} = \frac{GM R}{r^3} + \frac{1}{\rho}\frac{\partial p}{\partial R}$$

radial pressure much smaller than centrifugal force.

Compare centrifugal to pressure:

order unity

$$\frac{1/\rho\,\partial p/\partial R}{\frac{GM}{r^3}\cdot R} = \frac{c_s^2}{\frac{GM}{r^3}\cdot R^2}\cdot\frac{\partial \ln p}{\partial \ln R} = \frac{\cancel{\Omega^2} H^2}{\cancel{\Omega^2}\cdot R^2}\cdot\frac{\partial \ln p}{\partial \ln R} \approx \left(\frac{H}{R}\right)^2$$

Deviation is of order $O(h^2)$