The Shakura-Sunyaev model

Class 12 3/3/20

Turbulence is a source of viscosity

$$\rho \partial_t u_i = \rho F_i - \partial_j (p \delta_{ij} + \rho u_i u_j)$$

For mean quantities

Decompose in mean and fluctuation

$$\rho \partial_t (\bar{u}_i + u_i') = \rho F_i - \partial_j \left(\bar{p} \delta_{ij} + p' \delta_{ij} + \rho (\bar{u}_i + u_i')(\bar{u}_j + u_j') \right)$$

Average out $\langle u_i' \rangle = 0$

$$\rho \partial_t \bar{u}_i = \rho F_i - \partial_j \left(\bar{p} \delta_{ij} + \rho \bar{u}_i \bar{u}_j + \rho \langle u_i u_j \rangle \right) = 0$$

This term behaves like a viscous tensor, and transports angular momentum

Effectively, it's a diffusion of momentum, coming from the advection term, advecting momentum out of a gaussian surface.

Reynolds stress behave like a viscous term

$$\frac{\partial L}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R L u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu L \frac{d\ln\Omega}{d\ln R}\right)$$

$$\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\Sigma r^2 \Omega\right) u_r\right) = \frac{1}{2\pi r}\int \nabla T \, r\, d\phi$$

$$T = 2\pi r \; \nu \Sigma r \frac{d\Omega}{dr} \; r \qquad r \times F_v$$

$2\pi r$ → whole lenght

$\nu \Sigma r \frac{d\Omega}{dr}$: viscous force per length

r: arm

$$\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\Sigma r^2 \Omega\right) u_r\right) = \nabla \cdot \left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot v\right)\right]$$

$$\frac{\partial v_\phi}{\partial r} = \frac{d}{dr}(r\Omega) = \Omega + r\frac{d\Omega}{dr} =$$

Ω: pure rotation

$r\frac{d\Omega}{dr}$: shear

$$\nabla \cdot \left[\mu r \frac{d\Omega}{dr}\right]$$

Subs continuity equation on at momentum eq and set $\frac{\partial}{\partial \varphi}=0$

$$\frac{\partial L}{\partial t}+\frac{1}{R}\frac{\partial}{\partial R}\left(R L u_R\right)=\frac{1}{R}\frac{\partial}{\partial R}\left(\nu L \frac{d\ln\Omega}{d\ln R}\right)$$

$$\frac{\partial \overline{L_\phi}}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r L_\phi \bar{u}_r\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \rho \langle u'_\phi u'_r\rangle\right)$$

$$r^2 \rho \langle u'_r u'_\phi\rangle = \nu L \frac{d\ln\Omega}{d\ln R}$$

$$\nu = -\left(\frac{d\ln\Omega}{d\ln s}\right)^{-1}\frac{R^{\phi r}}{\rho\Omega} \; ; \; R^{\phi r}=\rho\langle u'_r u'_\phi\rangle$$

Equivalence turbulence and viscosity. This "momentum diffusion" is in fact a pressure. So, we can write

$$\boxed{R^{\phi r}=\alpha P}$$

The stress is proportional to the pressure. The causality is correct here. The stress generates a pressure

$$\nu=-\left(\frac{d\ln\Omega}{d\ln r}\right)\frac{\alpha P}{\rho\Omega}=\frac{2}{3}\alpha\frac{\cancel{\rho}\, c_s^2}{\cancel{\rho}\,\Omega}$$

$$\boxed{\nu \simeq \alpha c_s H}$$

Two different ways to write it. Both physical. The visco-

ν is prop to a lenght H is the sonic scale, the last isotropic length. The size of the eddies is H. The speeds are still thermal, so

$$[\nu] = LU \Rightarrow \nu = \alpha c_s H$$

If α is constant, it's possible to solve analytically for several disk phenomena

Time scales $\quad \tau = \frac{r^2}{\nu} = \left(\frac{H}{R}\right)^{-2} \frac{1}{\alpha \Omega}$

For $\left(\frac{H}{R}\right) \simeq 0.05 \quad \alpha \approx 10^{-2}$

$$\dot{m} = 3\pi \nu \Sigma = 3\pi \alpha c_s H \Sigma \qquad \alpha \approx 10^{-2}$$

Turbulence

what leads to turbulence – Rayleigh criterion

$$\frac{\partial u}{\partial t} + (u\,\nabla)u = -\frac{1}{\rho}\nabla p + g$$

Decompose the flow into base (time-indepent) and perturbation
$u(\vec{x},t) = \bar{u}(\vec{x}) + u'(\vec{x},t)$, with $\bar{u}_r = 0$ and $\bar{u}_\phi = \Omega r \gg u_\phi'$

$$\frac{\partial u_r}{\partial t} + u_\phi \frac{\partial}{\partial \phi} u_r - \frac{u_\phi^2}{r} = \Omega^2 r \xrightarrow{u=\bar{u}+u'} \frac{\partial u_r'}{\partial t} - \cancel{\Omega^2 r} - 2\Omega u_\phi = -\cancel{\Omega^2 r}$$

$$\boxed{\frac{\partial u_r'}{\partial t} - 2\Omega u_\phi = 0}$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial}{\partial r} u_\phi + \frac{u_\phi u_r}{r} = 0 \qquad \frac{\partial u_\phi'}{\partial t} + u_r' \frac{\partial}{\partial r}(\Omega r) + \frac{\Omega r u_r}{r}$$

$$\frac{\partial u_\phi'}{\partial t} + u_r'\left[\Omega + \frac{\Omega r}{\Omega}\frac{\partial \Omega}{\partial r}\right] + \Omega u_r \Rightarrow \boxed{\frac{\partial u_\phi'}{\partial t} + (2-q)\Omega u_r = 0}$$

$$q \equiv -\frac{d\ln\Omega}{d\ln r}$$

$$\Omega = \Omega_0\left(\frac{r_0}{r}\right)^q$$

The perturbation equations are thus

$$\partial_t u_r' - 2\Omega u_\phi' = 0 \quad \text{and} \quad \partial_t u_\phi' + \Omega(2-q)u_r' = 0$$

Now consider the perturbation as a Fourier mode $\psi' = \psi_0 e^{-i(\omega t - kx)}$

$$-i\omega\,\hat{u}_r - 2\Omega\,\hat{u}_\phi = 0 \quad \rightarrow \quad \hat{u}_\phi = -\frac{i\omega}{2\Omega}\hat{u}_r$$

$$-i\omega\,\hat{u}_\phi + \Omega(2-q)\,\hat{u}_r = 0$$

$$-i\omega\left(\frac{i\omega}{2\Omega}\right)\hat{v}_r + \Omega(2-q)\hat{v}_r = 0$$

$$-\frac{\omega^2}{2\Omega}\cancel{\hat{v}_r} + \Omega(2-q)\cancel{\hat{v}_r} = 0$$

$$\omega^2 - 2(2-q)\Omega^2 = 0$$

$$\boxed{\omega^2 = 2\Omega^2(2-q)}$$ if $q<2$, then the flow is stable

This frequency is called epicyclic frequency, $\kappa = \Omega\sqrt{2(2-q)}$

$q = -\frac{d\ln\Omega}{d\ln r} < 2$ is the case for keplerian Disks So, they are stable

$$\boxed{\left|\frac{d\ln\Omega}{d\ln r}\right| < 2 \quad \text{Stability}}$$

Equivalent statement the angular momentum must increase outward

$$L = r^2\Omega$$

$$\frac{d\,\Omega r^2}{dr} = r^2\frac{d\Omega}{dr} + 2\Omega r$$

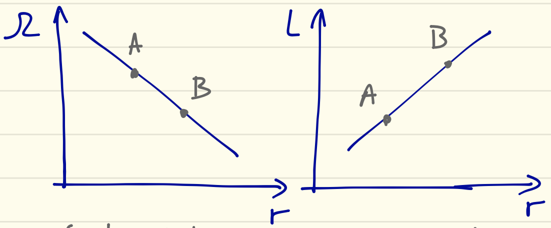
$$= r^2\frac{\Omega}{\Omega}\frac{r}{r}\frac{d\Omega}{dr} + 2\Omega r$$

$$= \frac{r^{\cancel{2}}\,\Omega}{\cancel{r}}\frac{d\ln\Omega}{d\ln r} + 2\Omega r$$

$$= \Omega r\left(2 + \frac{d\ln\Omega}{d\ln r}\right) = \Omega r(2-q)$$

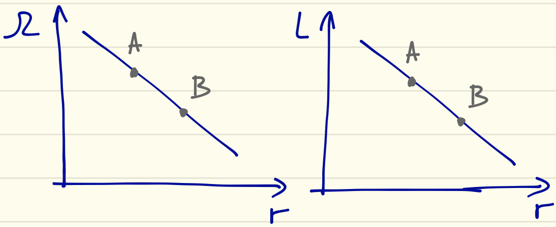
The sign of $\frac{dL}{dr}$ is the same sign of $(2-q)$ so, the condition $2-q>0$ for stability is equivalent to $dL/dr>0$, or L increasing with distance Let us understand it (show slides)

STABILITY



friction between A and B makes A slow down It loses (angular) momentum and jumps to an orbit of lower L, inwards

INSTABILITY



friction between A and B makes A slow down It loses (angular) momentum and jumps to an orbit of lower L, outwards

Since the orbit of lower angular momentum is outward, if A loses angular it must move outward B gaining angular momentum must move inward The rings swap and the situation is unstable