Class 17 2/27/20

Disk viscosity

Ok, we got the steady state solution, without radial velocities. Such a disk will just rotate and rotate without doing much. Show plot of time dissipation

The disks evolve in time! This is obvious at first, since the disk has to accrete to complete the formation of the star

It turns out angular momentum is a conserved quantity

$$\ell = R v_\phi = R^2 \Omega = \sqrt{G M_* R}$$

A gas parcel cannot simply "lose" angular momentum. It needs to be transported

Viscosity is something for engineers, but in accretion disk theory it occupies the central spot

Let's understand this process. The viscous force is

$$\frac{Du}{Dt} = -\nabla\phi - \frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla(2\nu\rho S) \qquad S = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

must be traceless

$\frac{1}{\rho}\nabla(2\nu\rho S)$ $\quad S = \frac{1}{2}\left(u_{i,j} + u_{j,i} - \frac{2}{3}\delta_{ij}\nabla\cdot u\right)$

Constant ν : $\frac{2\nu}{\rho}\nabla(\rho S) = \frac{2\nu}{\rho}\left[\rho\nabla\cdot S + S\cdot\nabla\rho\right]$

$= 2\nu\left[\nabla\cdot S + S\cdot\nabla\ln\rho\right]$ $\quad S = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

$= \nu\,\partial_j u_{i,j} + \nu\,\partial_j u_{j,i} + 2\nu S\nabla\ln\rho$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)$$

$$\nu\nabla^2 u + \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}u_j\right) + 2\nu S\nabla\ln\rho$$

$$= \nu\nabla^2 u + \frac{\nu}{3}\nabla(\nabla\cdot u) + 2\nu S\nabla\ln\rho$$

$$\frac{Du}{Dt} = -\nabla\phi - \frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla(2\nu\rho S)$$

$$r\phi = -\nu\left(r\frac{\partial^2\phi}{\partial r^2} + 3\frac{\partial\phi}{\partial r}\right)$$

This term will immediately lead to a radial velocity

Conservation form

$$\frac{\partial}{\partial t}(\phi) + \nabla\cdot(v\cdot\phi) = 0$$

quantity $\quad$ flux

$$\frac{\partial \rho}{\partial t} + \nabla(u\rho) = 0$$

Conservation of for momentum must be of form

$$\frac{\partial}{\partial t}(\text{momentum}) + \nabla\cdot(\text{momentum flux})$$

momentum : ρv ; flux $\rho u_i u_j$

$$\frac{\partial}{\partial t}(\rho u) = \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} = \rho\frac{\partial v}{\partial t} - u\cdot\nabla(u\rho) \qquad u_i\,\partial_j(u_j\rho)$$

$$\therefore \rho\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(\rho u) + u\,\nabla(u\rho)$$

$$\therefore \frac{\partial v}{\partial t} + (u\cdot\nabla)u = \rho\frac{\partial v}{\partial t} + \rho(u\cdot\nabla)u =$$

$$\frac{\partial}{\partial t}(\rho u) + u\,\nabla(u\rho) + \rho(u\cdot\nabla)u$$

$$\frac{\partial}{\partial t}(\rho u) + u_i\,\partial_j(u_j\rho) + \rho\, u_j\partial_j u_i = \partial_j(u_i u_j \rho)$$

$$\partial_j(u_i u_j \rho) = u_i\,\partial_j(u_j\rho) + (\rho u_j)\partial_j u_i$$

$$\frac{\partial}{\partial t}(\rho u_i) + \partial_j(u_i u_j \rho) =$$

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla\cdot[u \otimes \rho u]$$

$\rho v \bar{v}^2 n$

In cyl coordinates: $\qquad \nabla(u \cdot \rho) \to \nabla(u \cdot \Sigma) = \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma u_r)$

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \Sigma u_r)$$

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \Sigma u_r) = 0 \quad \text{(continuity)}$$

Momentum

$$\frac{\partial}{\partial t}(\rho u_i) + \partial_j(\rho u_i u_j) \Rightarrow \frac{\partial}{\partial t}(\Sigma u_i) + \partial_j(\Sigma u_i u_j) =$$

rad $\to$ $\frac{\partial}{\partial t}(\Sigma u_r) + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma u_r^2) = 0$

az $\to$ $\frac{\partial}{\partial t}(\Sigma u_\phi) + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma u_r u_\phi) = 0$

$l = r \cdot v_\phi \Rightarrow$

$$\therefore \frac{\partial}{\partial t}(\Sigma \Omega r^2) + \frac{1}{r}\frac{\partial}{\partial r}(\Sigma \Omega r^3 u_r) = \text{Viscous terms} = F(\nu)$$

$$\frac{\partial}{\partial t}(\text{ang mom}) + \nabla \cdot (\text{flux ang mom})$$

$\mu r \cdot \frac{\partial v_\phi}{\partial r}$

$$\frac{\partial}{\partial t}(\Sigma r^2 \Omega) + \frac{1}{r}\frac{\partial}{\partial r}\left(r(\Sigma r^2 \Omega) u_r\right) = \nabla \cdot \left[\mu r \frac{d\Omega}{dr}\right]$$

$\Sigma v_\phi = \Sigma r \Omega$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\mu r^2 \frac{d\Omega}{dr}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \varepsilon v_r\right) = 0 \qquad (r v_\phi)$$

$$\varepsilon\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r}\right) = \varepsilon \nabla(\mu \cdot s) \quad \times r$$

$$r v_\phi \frac{\partial \varepsilon}{\partial t} + r$$

→ don't integrate

$$\frac{\partial}{\partial t}\left(\Sigma r^2 \Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\Sigma r^2 \Omega\right) u_r\right) = \frac{1}{2\pi r}\int \nabla T \, r \, d\phi \qquad (\mathrm{II})$$

$$T = \underbrace{2\pi r}_{\text{whole lenght}} \; \underbrace{\nu \Sigma r \frac{d\Omega}{dr}}_{\text{viscous force per length}} \; \underbrace{r}_{\text{arm}}$$

$r \times F_v$

Both equations can be combined (substitute v_1 by $\partial\Sigma/\partial t$ in II)

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{(r^2\Omega)'}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \Omega'\right)\right]$$

For Keplerian disks

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\nu \Sigma r^{1/2}\right)\right]}$$

T

$$\frac{\partial}{\partial t}\left(\Sigma r^2 \Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\Sigma r^2 \Omega\right) u_r\right) = \nabla \cdot \left[\mu r \frac{d\Omega}{dr}\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\mu r^2 \frac{d\Omega}{dr}\right]$$

Amazingly enough, for $\nu \equiv$ cte this equation has an analytical solution (show)

One can also find the expression for the velocity, given $\frac{\partial \Sigma}{\partial t}$

$$v_r = \frac{-3}{\Sigma r^{1/2}} \frac{\partial}{\partial t} \left(\nu \Sigma r^{1/2} \right)$$

for constant Σ and ν, that yields

$$\boxed{u_r = -\frac{3\nu}{2r}}$$

And the mass accretion rate

$$\dot{m} = \oiint \Sigma u \hat{n} \, dA = -2\pi r \Sigma u_r$$

$$\boxed{\dot{m} = 3\pi \nu \Sigma}$$

One can also define the viscous timescale

$$[\nu] = \frac{cm^2}{s} \qquad \boxed{\tau_\nu = \frac{R^2}{\nu}}$$

These equations allows for comparisons to observations observations of T-Taur reveal mass accretion rates of

$$m \approx 10^{-8} \, M_\odot / yr$$

Using that and typical MMSN values ($\Sigma \approx 10^3\ g/cm^2$)

$$\nu \approx \frac{\dot{M}}{10\Sigma} \approx 10^{14}\ \frac{cm^2}{s}$$

Molecular viscosity:

$$\nu \equiv \lambda\ v_{TH}$$

$$\lambda = \frac{1}{n\,\sigma_{coll}} \qquad \left.\begin{aligned} n &\approx 10^{14}\ cm^{-3} \\ \sigma_{coll} &\approx 2\times 10^{-15}\ cm^{-2} \end{aligned}\right\} \lambda \approx 10\ cm$$

$v_{TH} \approx c_s \approx 1\ km/s$

$\therefore\ \nu \approx 10^7\ cm\ s^{-1}$

Seven orders of magnitude lower than required. Forming a star would take

$$\tau \approx \frac{R^2}{\nu} = \frac{(10^{14})^2}{10^7} \approx 10^{21}\,s \approx 10^{13}\ yrs.$$

Much longer than a Hubble time

Another mechanism, that acts as an effective viscosity must be involved