Math Methods in Physics I

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class #13

Row reduction: Gauss-Jordan elimination

$$ABC = (AB)C = A(BC)$$

$$BC_{kj} = \sum_l B_{kl} C_{lj}$$

$$\therefore [A(BC)]_{ij} = \sum_k A_{ik}(BC)_{kj} = A_{ik} B_{kl} C_{lj} = (ABC)_{ij}$$

orthogonal matrices: preserve vector length

$$Ar = v \qquad |r| = |v|$$

Property of orthogonal matrix $AA^T = 1 \quad \therefore A^T = A^{-1}$

In Complex space, if the length is to preserved, the property must be so that

$$Ar = v \qquad |r| = |v|$$

where $AA^{+} = 1 \quad \therefore A^{+} = A^{-1}$

And the dagger refers to transpose and complex conjugate.

Think of an example

Then show that $AA^T \neq 1$, but if we do both transpose and conjugate then it does. So

$AA^{+} = 1$ is more important than AA^T in complex space.

9.24
9.25

→ Gram-Schmidt method for finding orthonormal bases

A B C → Normalize A → $\hat{a}$
subtract B from $\hat{a}$ → $b = B - k\hat{a}$
Normalize b → $\hat{b}$
Subtract C from $\hat{b}$ and $\hat{a}$ → $c = C - k_1\hat{a} - k_2\hat{b}$
Normalize c → $\hat{c}$

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Eigenvalues and Eigenvectors

(self-value and self-vector), "characteristic values"

$r' = \lambda r$, or

$\boxed{Mr = \lambda r}$ a matrix that doesn't change the vector orientation.

what is the matrix that doesn't change the vector

$Mr = r$ (eigen-value of 1)

In general we will have

$Mr = \lambda r$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$$

$$(M-\lambda) r = 0$$

$\lambda = 1$
$\lambda = 6$

So, substituting λ,

$2x - y = 0$ for $\lambda = 1$
$x + y = 0$ for $\lambda = 6$

These vectors are the eigenvectors

Matrices in the Complex plane

Symmetric matrices $S^T = S$
Orthogonal matrices $SS^T = I$

In complex numbers:

Hermitian matrix $H^+ = H$
Unitary matrix $U^+ = U^{-1}$

The orthogonal matrix preserves length.

$$|Av| = |v|$$

Does the same apply in the complex plane? What kind of matrix preserves the norm?

$$|Av| = |v|$$

if A allows complex numbers. Let's see with $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Try

$$\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$ length $\sqrt{2}$

So

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

Check if $UU^T = I$ as in orthogonal matrices.

$$UU^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hm, close but no cigar. The last one is negative. We had

$$(-i)(-i) + (i)(i)$$

If we make both negative, it would work. We need to swap the sign of one of the factors in each term. That is, we need the conjugate.

$$UU^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So, multiply by

$$U\,(U^T)^*$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The important $(A^T)^*$ matrix is the transpose conjugate and denoted by

$$A^\dagger \quad (\text{dagger})$$

The all-important unitary matrix that preserves the norm of a vector in the complex plane has thus the property

$$|Uv| = |v|$$

$$UU^\dagger = I \quad \therefore \quad U^\dagger = U^{-1}$$

Notice that in the real plane this becomes again simply the orthogonal matrix

$$OO^T = I \quad \therefore \quad O^T = O^{-1}$$

The symmetric matrix $S^T = S$ has as counterpart in $\mathbb{C}$ the Hermitian matrix

$$H^\dagger = H$$

Real	Complex
Symmetric $S^T = S$	Hermitian $H^\dagger = H$
orthogonal $O^T = O^{-1}$	Unitary $U^\dagger = U^{-1}$

Diagonalizing matrices

The diagonal matrices is the matrix of eigenvalues

Exercises:

2-5

3-1

3.7.1 $a^2 - bc$

$b^2 - ac$

$c^2 - ab$

3.13

4.9

6.1 ⇒ Find AB, BA

6.6. Pauli matrices QM

6.7

6.8

6.9

6.32

4.12 ; 4.18 ; 4.19 ; 4.23(?) ;

Complex roots of polynomials with real coefficients come in pairs

The conjugate is a solution

Consider

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

Where all a_n are real

Suppose some complex number $\zeta = a + bi$ is a root of P

$$P(\zeta) = 0$$

We want to show that

$P(\bar{\zeta}) = 0$ as well, where $\bar{\zeta} = a - bi$ is the conjugate

$$a_0 + a_1 \zeta + a_2 \zeta^2 + \dots + a_n \zeta^n = 0$$

$$\Sigma a_n \zeta^n = 0$$

$$P(\bar{\zeta}) = \Sigma a_n (\bar{\zeta})^n = \Sigma a_n (\overline{\rho^n}) = \Sigma \overline{a_n \rho^n} = \overline{\Sigma a_n \rho^n} = 0$$

Thus $\bar{\zeta}$ is also a solution

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ $\quad$ $z_1 z_2 = (a+bi)(c+di) = ac - bd + i(ad + bc)$

$\overline{z_1 z_2} = (ac - bd) - i(bd + bc)$

$$z_1 z_2 = (a+bi)(c+di) = ac - bd + i(ad+bc)$$

$$\overline{z_1 z_2} = (ac-bd) - i(ad+bc)$$
$$\overline{z_1}\,\overline{z_2} = (a-bi)(c-di) = (ac-bd) - (ad+bc)$$

By induction, then $\overline{z}^{\,n} = \overline{z^n}$