Math Methods in Physics I

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Class #17

Eigendecomposition

orthogonal matrices : preserve vector length

$Ar = v \qquad |r| = |v| \Rightarrow r^T r = v^T v$

Property of orthogonal matrix $AA^T = I \quad \therefore A^T = A^{-1}$

$Ar = v \qquad |r| = |v|$

$(Ar) \cdot (Ar) = v \cdot v$

$(Ar)^T (Ar) = v^T v$

Product of the transpose: $(AB)^T = B^T A^T$

$A = A_{ij}$
$B = B_{ij}$

$c_{ij} = (AB)_{ij} = \sum_K A_{iK} B_{Kj}$

$c_{ji} = (AB)_{ji} = \sum_K B_{jK} A_{Ki}$

$AB = A_{iK} B_{Kj} = C_{ij}$

$(AB)^T = D_{ij} = C_{ji} = \sum_K A_{jK} B_{Ki} = B_{iK}{}^T A_{Kj}{}^T = B^T A^T$

So

$r^T A^T A r = v^T v$

for $r^T r = v^T v$, we must have $A^T A = I$

$A^T A = I \Leftrightarrow A^T = A^{-1}$

$$(AB)^T_{ik} = (AB)_{ki} = \sum_j A_{kj} B_{ji} = \sum_j A^T_{jk} B^T_{ij} = \sum_j B^T_{ij} A^T_{jk} = (B^T A^T)_{ik}$$

$$(AB)^T = B^T A^T$$

This is similar to the product of inverses. That leads to an interesting parallel with the same problem for inverse of matrices

$$(AB)^{-1} = B^{-1} A^{-1}$$

The order matter. To revert AB, you need to first undo B, then undo A. B: 2 steps forward, A turn left.

To revert $\overset{(AB)^{-1}}{(AB)}$: $\overset{B}{\text{2 steps forward}}$, $\overset{A}{\text{then turn left}}$, you must do: $\underset{A^{-1}}{\text{turn right}}$, then $\underset{B^{-1}}{\text{walk 2 steps backward}}$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Goincidences are suspicious... The transpose has something to do with the inverse...

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Eigenvectors and Eigenvalues (next page)

Eigenvalues and Eigenvectors

(self-value and self-vector), "characteristic values"

$r' = \lambda r$, or

$\boxed{Mr = \lambda r}$ a matrix that doesn't change the vector orientation.

what is the matrix that doesn't change the vector

$Mr = r$ (eigen-value of 1)

In general we will have

$$Mr = \lambda r$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$$

$$(M-\lambda)r = 0$$

$\lambda = 1$
$\lambda = 6$

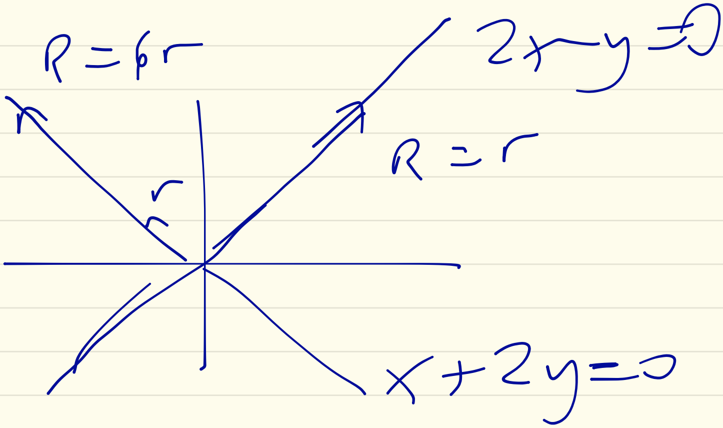
So, substituting λ,

$2x - y = 0$ for $\lambda = 1$
$x + y = 0$ for $\lambda = 6$

These vectors are the eigenvectors

$2x - y = 0 \quad \lambda = 1$

$x + 2y = 0 \quad \lambda = 6$



$$5x_1 - 2y_1 = x_1 \qquad 5x_2 - 2y_2 = 6x_2$$
$$-2x_1 + 2y_1 = y_1 \qquad -2x_2 + 2y_2 = 6y_2$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

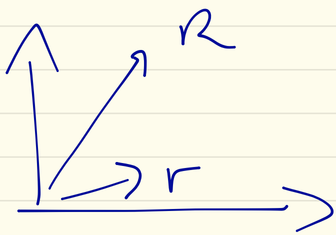
Make $r_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $r_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ unit vectors: $x_1 = 1/\sqrt{5}$; $y_1 = 2/\sqrt{5}$; $x_2 = -2/\sqrt{5}$; $y_2 = 1/\sqrt{5}$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Rotation   Deformation

$MC = CD$

$$\boxed{M = CDC^{-1}}$$



Works only for square matrices.

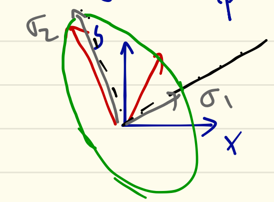
$Mr = CDC^{-1} r$

$$Mr = CDC^{-1}r$$

Rotate clockwise $(C^{-1}r)$
Stretch $D(C^{-1}r)$
Rotate counterclockwise to xy $(CDC^{-1}r)$

Mr

Show examples in python



Rotate to align with principle components

Meaning of transpose:

$$A = CDC^{-1}$$

$$A^T = (CDC^{-1})^T = (C^{-1})^T D^T C^T$$

C and C^{-1} are rotations. Transpose of rotation is its inverse $\therefore C^T = C^{-1}$
D is diagonal, so $D^2 = D$

$$A^T = {C^{-1}}^T D^T C^T = CDC^{-1}$$

$$A^T A = CD^2C^{-1}$$

$$M^T = V \Sigma U^T$$

$$M^T M = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$$ inverts rotation, keeps scaling

In general, if M is not square,

$$M = U \Sigma V^T$$

M is $m \times n$
U is $m \times m$
Σ is diagonal $m \times n$
V^T is $n \times n$

For any matrix A, $A^T A$ is square $\sum A_{ik} A_{ki} = A_{(n \times n)}$
Also symmetric

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

Symmetric matrix: eigenvectors are orthogonal

V = matrix of eigenvectors x_i of $A^T A$
Σ = diagonal matrix with eigenvalues σ_i of $A^T A$
U = normalize

$C = A^T A \qquad Cx = \sigma x \qquad r_i = \frac{Ax_i}{\sigma}$

U has columns r_i
V has columsn x
Σ has diagonal σ

$$\Sigma V^T = \sigma x$$

$$U \Sigma V^T = \frac{Ax_i}{\cancel{\sigma}} \cancel{\sigma} x_i = A$$

$$\Sigma V^T = \sigma x$$

$$U \Sigma V^T = \frac{Ax}{\cancel{\sigma}} \cancel{\sigma} x \Rightarrow x \cdot x = 1$$

$$U = \frac{Ax}{\sigma} = r$$

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Matrices in Complex space

In Complex space, what is the equivalent of the orthogonal matrix, that preserves length?

The orthogonal matrix preserves length.

$$|Av| = |v|$$

Does the same apply in the complex plane? What kind of matrix preserves the norm?

$$|Av| = |v|$$

if A allows complex numbers. Let's see with $v = \binom{1}{0}$

Try $\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ length $\sqrt{2}$

So

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

Check if $UU^T = I$ as in orthogonal matrices.

$$UU^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hm, close but no cigar. The last one is negative. We had

$$(-i)(-i) + (i)(i)$$

If we make both negative, it would work. We need to swap the sign of one of the factors in each term. That is, we need the conjugate.

$$UU^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So, multiply by

$$U\,(U^T)^*$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The important $(A^T)^*$ matrix is the transpose conjugate. and denoted by

$$A^\dagger \quad (\text{dagger})$$

In this case, the all-important matrix that preserves the norm has

$$|Uv| = |v|$$

has property

$$UU^\dagger = I \qquad \therefore \qquad U^\dagger = U^{-1}$$

Notice that in the real plane this becomes again simply the orthogonal matrix

$$OO^T = I \qquad \therefore \qquad O^T = O^{-1}$$

Is this general? The norm in the complex plane is

$$|v| = (a+bi)(c-bi) = (c^2+b^2)^{1/2} = v \cdot v^*$$

So,

$$|Ur| = |v| \quad ; \quad |r| = |v|$$

In matrix form $|v|$ = inner product : one has to be transposed

$$= [v^*]^T [v] = v^T v^*$$

So $|Ur| = |v|$ means

$$\left[[Ur]^*\right]^T Ur = [v^*]^T [v]$$

Write $\dagger = *T$:

$$[Ur]^\dagger Ur = v^\dagger v \quad \Rightarrow \quad r^\dagger U^\dagger U r = v^\dagger v \quad \therefore \quad U^\dagger U = I$$

The all-important unitary matrix that preserves the norm of a vector in the complex plane has thus the property

$$|Uv| = |v|$$

$$UU^\dagger = I \quad \therefore \quad U^\dagger = U^{-1}$$

Notice that in the real plane this becomes again simply the orthogonal matrix

$$OO^T = I \quad \therefore \quad O^T = O^{-1}$$

The symmetric matrix $S^T = S$ has as counterpart in $\mathbb{C}$ the Hermitian matrix

$$H^\dagger = H$$

Real	Complex
Symmetric $S^T = S$	Hermitian $H^\dagger = H$
orthogonal $O^T = O^{-1}$	Unitary $U^\dagger = U^{-1}$

For symmetric matrices the eigenvectors are orthogonal

$$A_{ij} x_j^1 = \lambda^1 x_i^1 \qquad A_{ij} x_j^2 = \lambda^2 x_i^2$$

$$\sigma_1 v_1 \cdot \sigma_2 v_2 = \sigma_1 \sigma_2 v_1^\dagger v_2 = 0$$

Diagonalize hermitian matrices

In real space $\quad M = CDC^{-1} \qquad MC = CD \qquad D = C^{-1} M C$

$$Hr = \lambda r$$

Complex conjugate $(Hr)^\dagger = (\lambda r)^\dagger$

$$r^\dagger H = \lambda^* r^\dagger$$

$$r^\dagger H r = \lambda^* r^\dagger r$$

$$r^\dagger \lambda r = \lambda^* r^\dagger r \quad \therefore \quad \lambda = \lambda^*$$

A Hermitian matrix has real eigenvalues

Eigenvalues of symmetric matrix are orthogonal

$$Sr_1 = \lambda_1 r_1 \qquad Sr_2 = \lambda_2 r_2 \qquad S = S^T$$

$$(r_2)\cdot(Sr_1) = (Sr_1)\cdot r_2$$

$$\lambda_1 r_2^T \cdot r_1 = (Sr_1)^T r_2 = r_1^T S^T r_2 = r_1^T \lambda_2 r_2 = \lambda_2 r_1^T \cdot r_2$$

$$\therefore \lambda_1 r_1 \cdot r_2 = \lambda_2 r_1 \cdot r_2$$

Unless $\lambda_1 = \lambda_2$, one must have $r_1 \cdot r_2 = 0$ $\quad$ r_1 and r_2 orthogonal.

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Hermitian matrices too

$$Hr_1 = \lambda r_1 \quad ; \quad Hr_2 = \lambda r_2$$

$$\vec{r}_2 \cdot H\vec{r}_1 = r_2^\dagger \lambda_1 r_1 = \lambda_1 r_2^\dagger r_1$$

$$= (H\vec{r}_1)^\dagger r_2 = r_1^\dagger H r_2 = \lambda_2 r_1^\dagger r_2$$

$r_1 = r_2$

$$r_1 \cdot r_2 = r_2 r_1 \Rightarrow r_1^\dagger r_2 = r_2^\dagger r_1$$

So, eigenvalues are real $D = D^\dagger$

$$D = U^{-1} M U$$

$$D^\dagger = U^\dagger M^\dagger U^{-1\dagger} = U^{-1} M^\dagger U$$

M must be Hermitian

For general matrices,

$$A = U \Sigma V^{\dagger}$$

-> Gram-Schmidt method for finding orthonormal bases

$A\ B\ C$ -> Normalize A -> $\hat{a}$

Subtract B from $\hat{a}$ -> $b = B - k\hat{a}$

Normalize b -> $\hat{b}$

Subtract C from $\hat{b}$ and $\hat{a}$ -> $c = C - k_1\hat{a} - k_2\hat{b}$

Normalize c -> $\hat{c}$

Complex roots of polynomials with real coefficients come in pairs

The conjugate is a solution

Consider

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

Where all a_n are real

Suppose some complex number $\rho = a + bi$ is a root of P

$$P(\rho) = 0$$

We want to show that

$P(\bar{\rho}) = 0$ as well, where $\bar{\rho} = a - bi$ is the conjugate

$$a_0 + a_1 \rho + a_2 \rho^2 + \dots + a_n \rho^n = 0$$

$$\Sigma a_n \rho^n = 0$$

$$P(\bar{\rho}) = \Sigma a_n (\bar{\rho})^n = \Sigma a_n (\overline{\rho^n}) = \Sigma \overline{a_n \rho^n} = \overline{\Sigma a_n \rho^n} = 0$$

Thus $\bar{\rho}$ is also a solution

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ $\quad$ $z_1 z_2 = (a+bi)(c+di) = ac - bd + i(ad + bc)$

$\overline{z_1 z_2} = (ac - bd) - i(bd + bc)$

$$z_1 z_2 = (a+bi)(c+di) = ac - bd + i(ad+bc)$$

$$\overline{z_1 z_2} = (ac-bd) - i(ad+bc)$$

$$\overline{z_1}\,\overline{z_2} = (a-bi)(c-di) = (ac-bd) - (ad+bc)$$

By induction, then $\bar{z}^n = \overline{z^n}$