Math Methods in Physics I

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class #7

Determinants look weird and cumbersome

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

weird and cumbersome

Example of determinant in physics

$$A \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{curl} \quad \nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

The cross-product is somehow connected with determinants.
Let's see if we can find some pattern...

$$A\times B = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

$$(A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2 + (A_1 B_2 - A_2 B_1)\hat{e}_3$$

$$\begin{aligned}
&= \varepsilon_a \quad \hat{e}_1 A_2 B_3 \\
&+ \varepsilon_b \quad \hat{e}_1 A_3 B_2 \\
&+ \varepsilon_c \quad \hat{e}_2 A_3 B_1 \\
&+ \varepsilon_d \quad \hat{e}_2 A_1 B_3 \\
&+ \varepsilon_e \quad \hat{e}_3 A_1 B_2 \\
&+ \varepsilon_f \quad \hat{e}_3 A_2 B_1
\end{aligned}$$

$$\varepsilon_a = \varepsilon_c = \varepsilon_e = 1$$
$$\varepsilon_b = \varepsilon_d = \varepsilon_f = -1$$

—//—

$$\begin{aligned}
&= \varepsilon_{123} \quad \hat{e}_1 A_2 B_3 \\
&+ \varepsilon_{132} \quad \hat{e}_1 A_3 B_2 \\
&+ \varepsilon_{231} \quad \hat{e}_2 A_3 B_1 \\
&+ \varepsilon_{213} \quad \hat{e}_2 A_1 B_3 \\
&+ \varepsilon_{312} \quad \hat{e}_3 A_1 B_2 \\
&+ \varepsilon_{321} \quad \hat{e}_3 A_2 B_1
\end{aligned}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$
$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

Define Levi-Civita symbol

$$\varepsilon_{ijk}$$

$$\boxed{A\times B = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\, \hat{e}_i\, A_j\, B_k}$$

Compact form

$$(A \times B)_i = \varepsilon_{ijk}\, \hat{e}_i\, A_j B_k$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \text{ even \# permutation} \\ 0 & \text{if any index repeated} \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (1,3,2), \text{ or } (2,1,3) \text{ odd \# permutation} \end{cases}$$

Odd number of permutation: only needs 1 permutation (in the case of 3 indices)

Even number of permutations: needs at least 2 permutations.

Examples:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$
$$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$$
$$\varepsilon_{111} = 0;\ \varepsilon_{112} = 0;\ \varepsilon_{331} = 0;\ \varepsilon_{323} = 0$$

Usefulness. Vector products

$$A \times B = \varepsilon_{ijk} A_j B_k$$

Curl : $(\nabla \times A)_i = \varepsilon_{ijk}\, \hat{e}_i\, \partial_j A_k$

Property :

Permutations $\Rightarrow$ $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jik}$

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \sum_{i=1}^{3} \varepsilon_{ijk}\,\varepsilon_{imn} =$$

$$\varepsilon_{1jk}\varepsilon_{1mn} + \varepsilon_{2jk}\varepsilon_{2mn} + \varepsilon_{3jk}\varepsilon_{3mn}$$

In 2D

$$\varepsilon_{ij}\,\varepsilon_{ik} = \varepsilon_{1j}\,\varepsilon_{1k} + \varepsilon_{2j}\,\varepsilon_{2k}$$

$$\varepsilon_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$j=1 = 0$ } $j = 2 \Rightarrow 1$ { $j=1 \Rightarrow 1$

$j=1=0$ | $k=2 \Rightarrow 1$ | $k=1 \Rightarrow 1$

if $j \neq k = 0$ if $j \neq k = 0$

Kronecker delta

$$\varepsilon_{ij}\,\varepsilon_{ik} = \varepsilon_{1j}\,\varepsilon_{1k} + \varepsilon_{2j}\,\varepsilon_{2k} = \delta_{jk}$$

$$\boxed{\varepsilon_{ij}\,\varepsilon_{ik} = \delta_{jk}}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

The Kronecker delta is the identity matrix.

$$a_i = \sum_j a_j \delta_{ij}$$

Inner product

$$a \cdot b = \sum a_i b_j \delta_{ij} = a_i b_i$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \varepsilon_{1jk}\varepsilon_{1mn} + \varepsilon_{2jk}\varepsilon_{2mn} + \varepsilon_{3jk}\varepsilon_{3mn}$$

if $j=1$, k must be 3 $\qquad \varepsilon_{213}\varepsilon_{2mn} + \varepsilon_{312}\varepsilon_{3mn}$

$$-\varepsilon_{2mn} + \varepsilon_{3mn}$$

$m=1, n=3 \qquad -\varepsilon_{213} + \cancel{\varepsilon_{313}} = 1 \qquad \delta_{jm}\delta_{kn}$

$m=3, n=1 \qquad -\varepsilon_{231} + \cancel{\varepsilon_{331}} = -1 \qquad -\delta_{jn}\delta_{km}$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\boxed{\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}$$

$$\varepsilon_{ijk}\varepsilon_{kpq} = \varepsilon_{ij1}\varepsilon_{1pq} + \varepsilon_{ij2}\varepsilon_{2pq} + \varepsilon_{ij3}\varepsilon_{3pq}$$

If $i=1$, $\qquad$ j must be 2 $\quad \varepsilon_{123}\varepsilon_{3pq}$

if $p=1$, q must be 2 $\quad \delta_{ip}\delta_{jq}$

if $p=2$, q must be 1 $\quad -\delta_{iq}\delta_{jp}$

$$\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Example

$$\nabla \times \nabla \times A = \nabla \times (\nabla \times A) = \nabla \times B \quad ; \quad B = \nabla \times A$$

$$(\nabla \times B)_i = \varepsilon_{ijk}\,\partial_j B_k$$

$$B_i = \varepsilon_{ijk}\,\partial_j A_k$$

$$\therefore \nabla \times \nabla \times A = \varepsilon_{ijk}\,\partial_j\left[\varepsilon_{kpq}\,\partial_p A_q\right]$$

$$= \varepsilon_{ijk}\,\varepsilon_{kpq}\,\partial_j\,\partial_p A_q$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\,\partial_j\,\partial_p A_q$$

$$= \partial_j\,\partial_i A_j - \partial_j\,\partial_j A_i$$

$$= \partial_i(\partial_j A_j) - \partial_{jj} A_i$$

$$= \nabla(\nabla \cdot A) - \nabla^2 A$$

$$A\times(B\times C) = \varepsilon_{ijk} A_j (B\times C)_k = \varepsilon_{ijk} A_j \varepsilon_{kpq} B_p C_q$$

$$= \varepsilon_{ijk}\varepsilon_{kpq} A_j B_p C_q$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) A_j B_p C_q$$

$$= A_j B_i C_j - A_j B_j C_i$$

$$= B_i (A_j C_j) - C_i (A_j B_j)$$

$$= B (A\cdot C) - C (A\cdot B)$$