Math Methods in Physics I

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Class #27

Odd / even transforms

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx - i\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx$$

For odd functions, $f(x)\cos(kx)$ is odd and the integral cancels

$$g(k) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx = -\frac{i}{\pi}\int_{0}^{\infty} f(x)\sin(kx)\,dx$$

$$f(x) = \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk = 2i\int_{0}^{\infty} g(k)\sin(kx)\,dk$$

Substitute $g(k)$ to remove the imaginary factor

$$f(x) = 2i\left(\frac{-i}{\pi}\right)\int_0^{\infty} \sin kx\, dk \int_0^{\infty} f(x)\sin kx\, dx$$

$$= \frac{2}{\pi}\int_0^{\infty} g(k)\sin kx\, dk$$

$$g(k) = \int_0^{\infty} f(x)\sin kx\, dx$$ for odd functions

For even functions, keep the co-sine

$$f(x) = \frac{2}{\pi}\int_0^{\infty} g(k)\cos kx\, dk$$

$$g(k) = \int_0^{\infty} f(x)\cos kx\, dx$$ for even functions

$$\rho(r,t) = A\, e^{i\psi(r,t)}$$

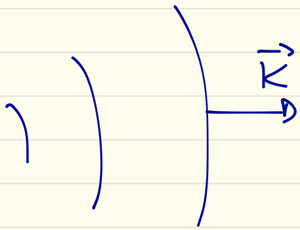
And we can define the phase

$$\psi = K \cdot r - \omega t$$

So that

$$K = \nabla\psi \quad \text{and} \quad \omega = -\partial_t \psi$$

The vector k is thus normal to surfaces of constant phase, or wavefronts.



Application to Poisson equation to the gravitational potential

$$g = -\frac{GM}{r^2} \qquad \oint g \cdot dA = -GM \int \frac{1}{r^2}\, dA = -4\pi GM$$

$$M = \int \rho\, dV \qquad \therefore \quad \oint g\, dA = -4\pi G \int \rho\, dV \qquad \int \nabla \cdot g\, dV = -4\pi G \int \rho\, dV$$

$$\nabla \cdot g = -4\pi G \rho \qquad \text{use} \quad g = -\nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho \qquad \text{set } 4\pi G = 1$$

$$\nabla^2 \phi = \rho$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \rho$$

Fourier transform $\quad \phi = \sum_n \phi_n e^{iK_n x} \quad \therefore \quad \frac{\partial}{\partial x}\phi = i \sum_n K_n \phi_n e^{iK_n x}$

$$\frac{\partial^2 \phi}{\partial x^2} = -\sum_n K_n^2 \phi_n e^{iK_n x} \quad \therefore$$

$$\sum_n \left(\frac{\partial^2}{\partial x^2} \hat{\phi} + \frac{\partial^2}{\partial y^2} \hat{\phi} + \frac{\partial^2}{\partial z^2} \hat{\phi} \right) = \sum_n \hat{\rho}$$

$$\sum_n \left(-K_x^2 - K_y^2 - K_z^2 \right) \hat{\phi} = \sum_n \hat{\rho}$$

$$-K^2 \hat{\phi} = \hat{\rho} \quad \therefore \quad \hat{\phi} = -\frac{\hat{\rho}}{K^2} \quad \therefore \quad \boxed{\phi = -F^{-1}\left(\frac{\hat{\rho}}{K^2} \right)}$$

Parseval theorem

Think of a vector V. As seen in coordinate system S with basis vector $\hat{e}_i$, it can be written

$$V = \sum_i V_i \hat{e}_i$$

where V_i are the components of V in S. As seen from another coordinate system S' with basis vectors $\hat{e}'_i$, it has a representation

$$V = \sum_i V'_i \hat{e}'_i$$

obviously the length of the vector is independent of the coordinate system used to represent it. In other words, we must have

$$\sum_i V_i^2 = \sum_i (V'_i)^2$$

Proceeding with the analogy, for a function $f(x)$ one can have a position space representation in δ-function basis as

$$f(x) = \int f'(x) \, \delta(x - x') \, dx$$

where the "component" of $f(x)$ along the "basis vector" $\delta(x-x')$ is $f(x')$ and we sum over all the possible axes. One can look at the same function in Fourier space representation as

$$f(x) = \int g(k) \, e^{-ikx} \, dk$$

where e^{-ikx} are the "basis vectors" and $g(k)$ are the components of $f(x)$. along these basis vectors. You would then agree that

$$\int |f(x)|^2 dx = \int |g(k)|^2 dk$$

So, Parceval theorem is just the restatement of the invariance of the length of a vector, independent of the representation used.

In our case it means that the energy in real space is equal to the energy in Fourier space.

Ptolemy. Great astronomer with wrong theory. Described the geocentric model nearly to perfection, thus able to predict correctly the position of the planets.

It's so good it's used today in planetaria all around the world to simulate planet motion. Gears and motors substitute epicycles.

Aristotle: Earth Air Water Fire

Seek natural places

Rain (water) falls from sky
Rocks (earth) fall when thrown
Smoke (air) rises
Flames (fire) rise.

Epicycles

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = g$$

$$\partial_t v_r' - 2\Omega v_\phi = 0$$

$$\partial_t v_\phi' + \Omega(2-q)v_r' = 0$$

$$\ddot{r} - r\dot{\phi}^2 = -\Omega^2 r$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

$$\begin{cases} \dfrac{\partial v_r}{\partial t} - \dfrac{v_\phi^2}{r} = -\Omega^2 r \\ \dfrac{\partial v_\phi}{\partial t} + \dfrac{v_r v_\phi}{r} = 0 \end{cases}$$

$$u_\phi = u_\phi' + \Omega r$$

$$\partial_t v_r' - (v_\phi' + \Omega r)^2 = -\Omega^2 r$$

$$\partial_t v_r' - 2\Omega v_\phi' = 0$$
$$\partial_t v_\phi' + 2\Omega v_r = 0$$

$$u_r' = \delta u_r e^{i(kr-\omega t)}$$
$$u_\phi' = \delta u_\phi e^{i(kr-\omega t)}$$

$$-i\omega u_r' - 2\Omega u_\phi' = 0$$
$$-i\omega u_\phi' + 2\Omega u_r' = 0$$

Solve : $\omega^2 = 4\Omega^2 = \kappa^2$ $\boxed{\kappa = 2\Omega}$

$\kappa = 2\Omega$ epycicylc frequency.