Math Methods in Physics I

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Class #20

Lagrange multiplier ✓ minimization example

Leibniz rule ✓ integral example (do Wolfram alpha first)

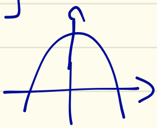
Multiple integral - Angle surface sphere geodesic?

Vector analysis ? Stokes theorem, Gauss theorem?

Path integral

Lagrange multipliers

$y = 1 - x^2$. Shortest distance between that and origin. That is, minimize $r^2 = x^2 + y^2$ given $y = 1 - x^2$.



Put circle centered at origin with $r = 1$.

Minimum of $f = x^2 + y^2$ given $y = 1 - x^2$

$$f = x^2 + (1 - x^2)^2 = x^4 - x^2 + 1$$

$$\frac{df}{dx} = 4x^3 - 2x = 0 \qquad x = 0 \text{ or } x = \pm\sqrt{1/2} \qquad (\text{also } \pm\infty)$$

Implicit differentiation

given $x = y + e^y$, find $\frac{dy}{dx}$

$\left(\frac{d}{dx}\right) \quad x = y + e^y \qquad 1 = \frac{dy}{dx} + e^y \frac{dy}{dx} = \frac{dy}{dx}\left(1+e^y\right)$

$$\frac{dy}{dx} = \frac{1}{1+e^y}$$

-> Applied to the problem

$f = x^2 + y^2 \quad ; y = 1 - x^2$

$df = 2x\,dx + 2y\,dy \quad : \quad \frac{df}{dx} = 2x + 2y\frac{dy}{dx}$

but $y = 1 - x^2$, so $\frac{dy}{dx} = -2x$

$x = 0$

$\frac{df}{dx} = 2x - 4xy \quad \Rightarrow \cancel{2x} - \cancel{4x}y = 0 \qquad y = 1/2 \;;\; x = \pm 1/2$

Lagrange multiplier

$\phi(x,y) = \text{constant} \quad (\text{constraint}) \qquad d\phi = 0$

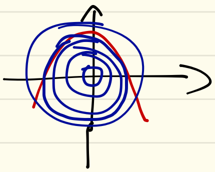
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

Define

$$\mathcal{L} = f + \lambda \phi$$

f cannot be increasing in the direction where $\phi = \text{const}$



$$f = x^2 + y^2$$

$$\phi = y - 1 + x^2 = 0$$

The minimum or max lies along the line

$$\nabla f = \lambda \nabla \phi$$

So, we can define the curve

$\mathcal{L} = f + \lambda \phi$ and find $\nabla \mathcal{L} = 0$

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} \right) dy = 0$$

pick λ so that $\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$

The $\lambda = - {}^{\partial f/\partial y}/{}_{\partial \phi/\partial y}$

$$\mathcal{L} = x^2 + y^2 + \lambda (y + x^2)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda \cdot 2x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 2y + \lambda = 0 \end{cases}$$

Solve with $y = 1 - x^2$

From 1^{st}, either $x=0$ or $\lambda=-1$. If $x=0$, $\lambda=-2$.

If $\lambda=-1$, the 2^{nd} eq gives $y=1/2$ and $x=\pm 1/\sqrt{2}$

Leibniz rule ✓ integral example (do Wolfram alpha first)

Do the integral

$$\int_0^\infty t^{2n+1} e^{-kt^2} dt = \frac{n!}{2k^{n+1}}$$

$$f(x) = \frac{dF(x)}{dx} \rightarrow \int_a^x f(t)\,dt = F(t)\Big|_a^x = F(x) - F(a)$$

$$\therefore \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}F(x) = f(x)$$

Similarly

$$\frac{d}{dx}\int_x^a f(t)\,dt = \frac{d}{dx}\left[F(t)\Big|_x^a\right] = \frac{d}{dx}\left[F(a) - F(x)\right] = -f(x)$$

In general:

$$I = \int_{u(x)}^{v(x)} f(x,t)\,dt$$

$$\frac{dI}{dx} = \frac{\partial I}{\partial u}\frac{du}{dx} + \frac{\partial I}{\partial v}\frac{dv}{dx} + \frac{\partial I}{\partial x}$$

$$\frac{\partial I}{\partial v} = f(v)$$

$$\frac{\partial I}{\partial u} = -f(u)$$

Leibniz rule

So $$\boxed{\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx} + \int_u^v \frac{\partial f}{\partial x}\,dt}$$

$$\frac{dI}{dx} \text{ for } I = \int_a^b f(x,t)\,dt$$

$$\frac{dI}{dx} = \frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

Useful for calculating integrals

Ex: Calculate

$$\int_0^\infty t^{(2n+1)} e^{-kt^2}\,dt$$

Show Wolfram example

$$\Gamma(n) = (n-1)! \qquad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$$

Function that interpolates factorial from integer to real.

Ex:

$$I = \int_0^\infty t\, e^{-Kt^2}\, dt = -\frac{1}{2k} e^{-Kt^2}\Big|_0^\infty = \frac{1}{2K}$$

Now calculate multiple derivatives

$$\frac{dI}{dK} = \frac{d}{dK}\int_0^\infty t\, e^{-Kt^2}\, dt = \int_0^\infty t \frac{d}{dK} e^{-Kt^2}\, dt = \int_0^\infty -t^3 e^{-Kt^2}\, dt$$

$$= -\frac{1}{2K^2}$$

So $\int t^3 e^{-Kt^2} dt = \frac{1}{2K^2}$

Repeat

$$\frac{d^2 I}{dK^2} = \frac{d}{dK}\int_0^\infty -t^3 e^{-Kt^2}\, dt = \int_0^\infty t^5 e^{-Kt^2}\, dt = -\frac{(2)}{2K3} = \frac{1}{K^3}$$

$$\frac{d^3 I}{dK^2} = \frac{d}{dK}\int_0^\infty t^5 e^{-Kt^2}\, dt = \int_0^\infty -t^7 e^{-Kt^2}\, dt = -\frac{3}{K4}$$

So, $\int t^7 e^{-Kt^2} dt = \frac{3}{K4}$

$$\int_0^\infty t^{(2n+1)} e^{-Kt^2}\, dt = \frac{n!}{2K^{(n+1)}}$$

surface sphere geodesic?

Sphere coordinates Example airplane

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \qquad ds^2 = ds\cdot ds$$

On the sphere, $dr=0$, use $r=1$

$$ds^2 = d\theta^2 + \sin^2\theta\,d\phi^2$$

Length $L = \int_A^B |ds| = \int_A^B \sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}$

$$d\theta^2 + \sin^2\theta\,d\phi^2 = d\theta^2\left(1 + \sin^2\theta\,\frac{d\phi^2}{d\theta^2}\right)$$
$$= d\theta^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$\theta = \theta(t)$; $\phi(t)$

$$ds = r\sqrt{\theta'^2 + \phi'^2\sin^2\theta}\,dt$$

$$S = \int_A^B \sqrt{\theta'^2 + \phi'^2\sin^2\theta}\,dt$$

$$ds^2 = d\theta^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$$L = \int_A^B |ds| = \int_A^B \sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

$$d\theta^2 + \sin^2\theta\, d\phi^2 = d\theta^2\left(1 + \sin^2\theta\, \frac{d\phi^2}{d\theta^2}\right)$$

$$= d\theta^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$$L = \int_A^B |d\theta|$$

$$s = \int |\theta'(t)|\, dt = |\theta(b) - \theta(a)|$$

Minimized if and only if $\phi' = 0$. Thus lie on a meridian of the sphere $\phi = \phi_0 \equiv \text{const}$. In cartesian,

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{cte} \equiv \phi_0$$

$$\frac{y}{x} = \tan\phi_0 \qquad y\cos\phi_0 = x\sin\phi_0$$

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Path integral

Stokes theorem, Gauss theorem?

Work

$$W = F \cdot d$$

infinitesimal

$$dW = F \cdot dr \quad ; \quad W = \int F \cdot dr$$