Math Methods in Physics I

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Class # 3

Binomials

$$\binom{p}{n} = \frac{p!}{n!\,(p-n)!} \quad \text{for } 0 \le n \le p$$

$$y(x) = (1+x)^p$$

$$y(x) = \sum_{n=0}^{\infty} \left[\frac{d^{(n)}}{dx^{(n)}} (1+x)^p \right] \Bigg|_{x=0} \frac{x^n}{n!}$$

$$\frac{d}{dx}(1+x)^p = p(1+x)^{p-1}$$

$$\frac{d^2}{dx^2}(1+x)^p = p(p-1)(1+x)^{p-2}$$

$$\frac{d^3}{dx^3}(1+x)^p = p(p-1)(p-2)\,(1+x)^{p-3}$$

$$\frac{d^{(n)}}{dx^{(n)}}(1+x)^p = \frac{p!}{(p-n)!}\,(1+x)^{p-n}$$

Evaluated at zero,

$$\frac{d^{(n)}}{dx^{(n)}}(1+x)^p \Bigg|_{x=0} = \frac{p!}{(p-n)!}$$

$$\boxed{f(x) = (1+x)^p = \sum_{n=0}^{\infty} \left[\frac{p!}{(p-n)!\, n!}\right] x^n = \sum_{n=0}^{\infty} \binom{p}{n} x^n}$$

useful as $\frac{1}{1+x} = (1+x)^{-1}$ or binomial with $p = -1$

$$\frac{1}{1+x} = \sum \binom{p}{n} x^n = 1 - x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots$$

$$= 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

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integrate $\int \frac{1}{1+x}\, dx = \ln(1+x)$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

So,

$$\ln(x) = \ln(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

Series for π.

$$\operatorname{tg}\left(\frac{\pi}{4}\right) = 1 \quad \therefore \quad \arctan 1 = \frac{\pi}{4} \quad \therefore \quad \boxed{\pi = 4 \cdot \arctan 1}$$

You may remember that

$$\arctan(x) = \int \frac{dx}{1+x^2}$$

$$\frac{1}{1+t^2} = (1+t^2)^{-1} = (1+y)^{-1} \quad ; y = t^2$$

Expand binomial

$$(1+y)^{-1} = 1 - y + y^2 - y^3 + y^4$$

$$\therefore \quad (1+x^2)^2 = 1 - x^2 + x^4 - x^6 + \dots$$

$$\arctan(x) = \int (1+x^2)^{-1} dx = \int dx - \int x^2 dx + \int x^4 dx - \int x^6 dx$$

$$= 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\therefore \quad \pi = 4\left(1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$$

Long division

- Divide 1st term of numerator by 1st term of denominator, put in answer
- Multiply denominator by that answer, put below numerator
- Subtract to create new polynomial
- Repeat.

Example $\dfrac{x^2-3x-10}{x+2}$

$$\begin{array}{r|l} & x \; - 5 \\ \hline x+2 & x^2-3x-10 \\ & -(x^2+2x) \\ \hline & \quad -5\cancel{x} -10 \\ & -(5x+10) \\ \hline & \qquad\qquad 0 \end{array}$$

Good! $(x^2-3x-10)=(x-5)(x+2)$

Another use. Expand $\dfrac{1}{\cos x}$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} - \ldots$

Remind them of trick to remember: small angles (plus application in physics). $\sin x \approx x$

$$\begin{array}{r|l} & 1 \\ \hline 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots & 1 \\ & -\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\right) \\ \hline & \frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots \end{array}$$

$$1 + \frac{x^2}{2!} - \frac{5}{24}x^4 + \frac{61}{720}x^6 - \dots$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \;\Big)\; 1$$

$$-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)$$

$$\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots$$

$$-\left(\frac{x^2}{2!} - \frac{x^4}{2!\,2!} + \frac{x^6}{4!\,2!} - \dots\right)$$

$$\frac{5}{24}x^4 - \frac{7x^6}{360}$$

$$-\left(\frac{5}{24}x^4 - \frac{5x^6}{48}\right)$$

$$-\frac{x^4}{4!} + \frac{x^4}{2!\,2!} = -\frac{x^n}{n!} + \frac{x^n}{2\cdot(n-2)!}$$

$$= x^n\left(\frac{1}{2(n-2)!} - \frac{1}{n!}\right) = x^n\left(\frac{n! - 2!(n-2)!}{n!\,2!(n-2)!}\right)$$

$$n=4 \Rightarrow \frac{4! - 4}{4!\cdot 4} = \frac{4(3!-1)}{4\cdot 4!} = \frac{5}{24}$$

$$n=6 \Rightarrow \frac{6! - 2\cdot 4!}{6!\cdot 2\cdot 4!} = \frac{6\cdot 5\cdot 4! - 2\cdot 4!}{6!\cdot 2!\cdot 4!} = \frac{\cancel{4!}\cdot \cancel{28}^{\,14}}{\cancel{4!}\cdot \cancel{2!}\,6!} = \frac{14}{720}$$

$$tg(x) = \frac{\sin x}{\cos x}$$

$$\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \times \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots\right)$$

$$x + \frac{x^3}{2} + \frac{5x^5}{24} - \frac{x^3}{3!} - \frac{x^5}{2 \cdot 3!} + \frac{x^5}{5!} + \dots$$

$$\frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$x + \frac{x^3}{3} + \frac{16}{120} x^5 \rightarrow \frac{4 \cdot \cancel{4}}{30 \cdot \cancel{4}} \left(\frac{2}{15}\right)$$

$$\boxed{tg \approx x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots}$$

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Approximate up to three terms

$$(x+1)\sin x = (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$= x + x^2 - \frac{x^3}{3!} + O(x^4)$$

$$e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots\right) \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \dots$$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \dots$$

$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

$$e^{\tan x} = 1 + \tan x + \frac{(\tan x)^2}{2} + \frac{(\tan x)^3}{3!} + \dots$$

$$= 1 + \left(x + \frac{x^3}{3}\right) + \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2 + \frac{x^3}{6}$$

$$= 1 + x + \frac{x^2}{2} + \left(\frac{x^3}{3} + \frac{x^3}{6}\right) + \dots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \dots$$ find x^4 term

Use of series: Mostly to approximate complicated functions.

$$f(x) = \ln\left(\frac{(1+x)}{(1-x)}\right)^{1/2} - \tan(x)$$

$$\ln\left(\frac{1+x}{1-x}\right)^{1/2} = \int \frac{dx}{(1-x)^2} \qquad \frac{d}{dx}\ln\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{1\cancel{(1-x)}}{2(1+x)}\left[\frac{(1-x)+(1+x)}{(1-x)^{\cancel{2}}}\right]$$

$$= \frac{2}{(1-x)^2}$$

$\therefore$ Integrate this binomial

$$\int \frac{dx}{(1-x)^2} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

$$\tan = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$$

$$f(x) = \frac{x^5}{15} + \frac{4x^7}{45} \approx \frac{x^5}{15} \text{ for small } x$$

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$$\frac{d^5}{dx^5}\left(\frac{1}{x}\sin x^2\right)\Bigg|_{=0}$$