Math Methods in Physics I

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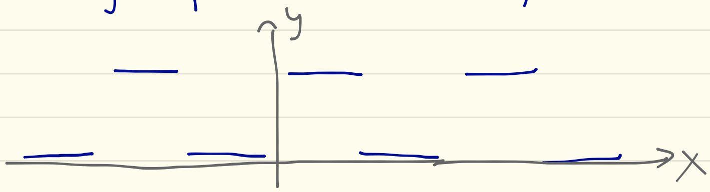
Class #23

Dirichilet Conditions

Convergence of fourier series. Box function



Value at $x=0$ where $f(x)$ goes from 0 to 1?

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

The value of the series at $x=0$ is $1/2$, but what does it have to do with the function?

A well-behaved function for Fourier series:

The Fourier series converges to the value of the function while it is continuous and to the midpoint of the jump in discontinuities if

- $f(x)$ is periodic
- single-valued in the period
- finite number of maximum and minimum values
- finite number of discontinuities
- $\int_{-\pi}^{\pi} |f(x)|\, dx$ is finite

Single-valued. One value of $f(x)$ for each x. $x^2+y^2=1$. Not single valued.

Infinite number of maxima and minima: $\sin(1/x)$. Oscillates infinite times as $x \to 0$ (show graph)

Make a function $f(x)=1$ for $\sin(1/x)>0$
$f(x)=-1$ for $\sin(1/x)<0$

This function has an infinite number of discontinuities.

$$y = 1/x \to \int_{-\pi}^{\pi} \left|\frac{1}{x}\right| dx = 2\int_0^{\pi} \frac{1}{x}\, dx = 2 \ln x \Big|_0^{1} = \infty$$

Ruled out by Dirichlet

Yet $y = 1/\sqrt{x}$

$$\int_{-\pi}^{\pi} \frac{1}{\sqrt{x}}\, dx = 2\int_0^{\pi} \frac{dx}{\sqrt{x}} = 4\sqrt{x}\Big|_0^{\pi} = 4\sqrt{\pi}$$

So a periodic function $1/\sqrt{x}$ between $-\pi$ and π can be expanded in Fourier series. Most times you don't need to evaluate the integral. Simply check if the function is bounded. But $1/\sqrt{x}$ is not bounded and still converges. why?

<u>Gibbs phenomenon</u> - The wiggling won't go away. It won't satisfy the Dirichlet conditions, will it? It doesn't converge to either the value or the half of it.

Summing numerical series: $f(x)$ converges to $1/2$ at $x=0$

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) \quad \text{since } \sin 0 = 0 \text{ and } \cos 0 = 1$$

Thus

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Complex form of Fourier series

If

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + \dots$$

Write

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i} \qquad \cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

So, we can find the complex form of Fourier series

$$f(x) = a_0 + \frac{a_1}{2}\left(e^{ix} + e^{-ix}\right) + \frac{a_2}{2}\left(e^{i2x} + e^{-i2x}\right) + \frac{b_1}{2i}\left(e^{ix} - e^{-ix}\right) + \frac{b_2}{2i}\left(e^{i2x} - e^{i2x}\right)$$

$$= a_0 + \frac{1}{2}(a_1 - ib_1)e^{ix} + \frac{1}{2}(a_1 + ib_1)e^{-ix} + \frac{1}{2}(a_2 - ib_2)e^{i2x} + \frac{1}{2}(a_2 + ib_2)e^{-i2x} + \dots$$

$$= c_0 + c_1 e^{ix} + c_{-1}e^{-ix} + c_2 e^{i2x} + c_{-2}e^{-i2x} + \dots$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Find c_n's (average value of e^{ikx} cancels when $k \in \mathbb{Z}^*$)

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Find c_n: Multiply by e^{-inx} and again find average value

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = c_0\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}dx + c_1\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{ix}dx$$

$$+c_{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{ix}dx + \dots$$

All zero, except the one containing $K=0$, i.e, the c_n term. So,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = c_n\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{inx}dx = c_n\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}dx = c_n$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

For real $f(x)$, $c_{-n} = \bar{c}_n$

Again, expand the box function

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}\cdot 0\cdot dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\cdot 1\cdot dx$$

$$=\frac{1}{2\pi}\frac{e^{-inx}}{-in}\Big|_0^{\pi} = \frac{1}{-2\pi in}\left(e^{-in\pi}-1\right) = \begin{cases}\frac{1}{\pi in}; & n \text{ odd}\\ 0\,; & n \text{ even} \neq 0\end{cases}$$

$$c_0 = \frac{1}{2\pi}\int_0^{\pi}dx = \frac{1}{2}$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{1}{2} + \frac{1}{i\pi}\left(e^{ix} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \dots\right)$$

$$+\frac{1}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + \dots\right)$$

Can leave it like this, or collect the terms

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{e^{ix}-e^{-ix}}{2i} + \frac{1}{3}\frac{e^{3ix}-e^{-3ix}}{2i} + \dots\right)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \dots\right)$$

Other intervals

We've been considering $-\pi, \pi$ as the interval of length 2π. Consider intervals of length $2L$:
The function $\sin\left(\frac{n\pi x}{L}\right)$ has period $2L$ $\because$

$$\sin \frac{n\pi}{L}(x+2L) = \sin\left(\frac{n\pi x}{L} + 2\pi n\right) = \sin \frac{\pi x x}{L}$$

$\cos\left(\frac{n\pi x}{L}\right)$ and $e^{in\pi x/L}$ have period $2L$. The coefficients and functions then would be

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + a_2 \cos\frac{2\pi x}{L} + \dots$$

$$+ b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + \dots$$

$$= \frac{a_0}{2} + \sum_1^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$$

The integral would be over a period; so, we replace

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{ replace by } \frac{1}{2L}\int_{-L}^{L}$$

So

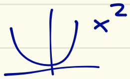
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

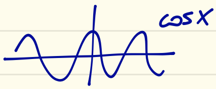
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad \sin x \to \sin\frac{2\pi x}{L} \quad \text{normalize period}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

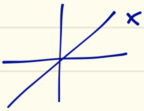
Even and odd functions

Even $\quad f(-x) = f(x)$





Odd $\quad f(-x) = -f(x)$



Even × Even = Even
Odd × Odd = Even
Odd × Even = Odd

Any function can be written in terms of a sum of an even and an odd function

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{\text{even}} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{\text{odd}}$$

swap x for $-x$

Example:

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right) = \cosh x + \sinh x$$

Change for (i) -> conjugate? Give as extra example?

$$e^{ix} = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) + \frac{1}{2}\left(e^{ix} - e^{-ix}\right)$$? Need to conjugate?

Integrals of even/odd functions over the interval

$$\int_{-L}^{L} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^L f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$$

Coefficients simplify. If $f(x)$ is odd. Sines are odd and cosines even. So,

$f(x)\sin\left(\frac{n\pi x}{L}\right)$ is even

$f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd

So, a_n is integral of odd function. a_n are all zero. b_n is even, twice the integral over half-period

$$f(x) \text{ for odd } \begin{cases} b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \\ a_n = 0 \end{cases}$$

$f(x)$ odd is expanded in sine series

If $f(x)$ is even, $f(x)\sin$ is odd, $f(x)\cos x$ is even, so the sines will cancel in integration, and

$$f(x) \text{ even } \begin{cases} a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = 0 \end{cases}$$

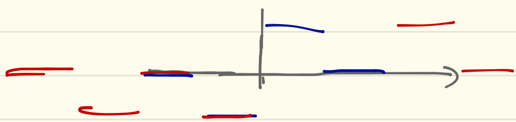
$f(x)$ even is expanded on cosine series

Do the example from book

$$f(x) = \begin{cases} 1 & ; \ 0 < x < 1/2 \\ 0 & ; \ 1/2 < x < 1 \end{cases}$$

In sine series/ cosine series / exponential

Sine series. Make it odd



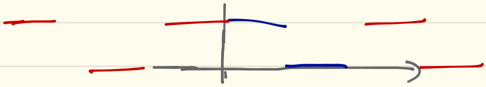
The period is 2. Odd function. $a_n = 0$

$$b_n = 2\int_0^1 f(x)\sin n x\pi \, dx = 2\int_0^{1/2} \sin n\pi x \, dx = \frac{-2}{n\pi}\cos n\pi x \Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$$

$b_1 = \frac{2}{\pi}$; $b_2 = \frac{4}{2\pi}$; $b_3 = \frac{2}{3\pi}$; $b_4 = 0$, ...

$$f(x) = \frac{2}{\pi}\left(\sin\pi x + \sin 2\pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{\sin 6\pi x}{3} + \dots\right)$$

Sketch in cosine series. Make it even



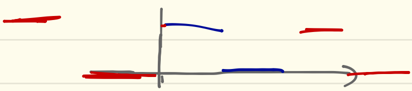
All $b_n = 0$; period 2

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = \frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\pi x - \frac{\cos 3\pi x}{3} + \frac{\cos 5\pi x}{5} + \cdots\right)$$

Do the general exponential. Just repeat



Period 1

$$c_n = \int_0^1 f(x)\, e^{-2in\pi x}\,dx = \int_0^{1/2} e^{-2in\pi x}\,dx$$

$$= \frac{1 - e^{-in\pi}}{2in\pi} = \frac{1-(-1)^n}{2in\pi} = \begin{cases} \frac{1}{in\pi} ; & n \text{ odd} \\ 0 ; & n \text{ even} \neq 0 \end{cases}$$

$$C_0 = \int_0^{1/2} dx = \frac{1}{2}$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[e^{2i\pi x} - e^{-2i\pi x} + \frac{1}{3} e^{6i\pi x} - \frac{1}{3} e^{-6\pi i x} + \dots \right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin 2\pi x + \frac{\sin 6\pi x}{3} + \dots \right]$$