Math Methods I – Class 2

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Sep 1st, 2016

Class #2

Properties of series

Geometric progression. Consider the series

$$S = a + ar + ar^2 + ar^3 + \dots$$

Multiply it by r and subtract the result from the original

$$\begin{aligned} S &= a(1 + r + r^2 + r^3 + \dots) \\ - rS &= a(\quad r + r^2 + r^3 + \dots) \end{aligned}$$

$$S(1-r) = a \quad \therefore S = \frac{a}{1-r}$$

The value of S we found is the sum of the series.

- Convergent series: has finite sum
- Divergent series has infinite sum

Up to the n^{th} term

$$S_n = a(1 + r + r^2 + r^3 + \dots + r^{n-1})$$

and repeating the same trick,

$$\boxed{S_n = \frac{a(1-r^n)}{(1-r)}}$$

One can see from this that if $r>1$ the series diverges. If $r<1$ the series converges. The series sum S is the limit of S_n as n goes to infinity

$$\boxed{S = \lim_{n\to\infty} S_n = \frac{a}{1-r}}$$

It is important to know if a series diverges

Weird things can happen if you apply ordinary algebra to divergent series. For instance

$$S = 1+2+3+4+5+\dots$$
$$4S = 4+8+12+16+20+\dots$$

$$\begin{aligned} S &= 1+2+3+4+5+6+\dots \\ 4S &= \quad\;\; 4 \quad\;\; + 8 \quad\;\; + 12 + \dots \\ \hline -3S &= 1-2+3-4+5-6 \end{aligned}$$

this is the series expansion of $\frac{1}{(1+x)^2}$ for $x=1$

$$\therefore -3S = \frac{1}{(2)^2} = \frac{1}{4} \qquad \therefore S = -\frac{1}{12}$$

$$1+2+3+4+5+\dots = -\frac{1}{12}$$

That is, we conclude that the sum of all infinite natural numbers $\sum_{n=1}^{\infty} n = -\frac{1}{12}$, which is nonsense.

Another example, from your book

$$S = 1 + 2 + 4 + 8 + 16$$
$$2S = 2 + 4 + 8 + 16 + 32$$

$$\therefore 2S - S = \cancel{2} + \cancel{4} + \cancel{8} + \cancel{16} + \cancel{32} + \dots$$
$$-1 - \cancel{2} - \cancel{4} - \cancel{8} - \cancel{16} - \dots$$

$\therefore \boxed{S = -1}$ Nonsense.

These spuriously nonsensical values appear simply because, doing algebra on divergent series, we are subtracting infinities from one another, which is an indeterminate operation.

Statements about convergence

$\lim_{n\to\infty} S_n = S$; where S is finite. Then the series is convergent.

S is the sum of the series.

The difference $R_N = S - S_n$ is called the remainder.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (S - S_n) = S - \lim_{n\to\infty} S_n = S - S = 0$$

Convergence tests:

Simple tests:

If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges

Notice that the opposite is not true. Some series that have $a_\infty \to 0$ diverge!

Comparison test: Let

Let $m_1 + m_2 + m_3 + \dots$ convergent

If $S = a_1 + a_2 + a_3$

has $|a_n| \leq m_n$, then S converges.

If $d_1 + d_2 + d_3 + \dots$ diverges, then

$|a_1| + |a_2| + |a_3|$

diverges if $|a_n| \geq d_n$ from any point on.

Example: test $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$

Compare with geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

all terms
$$\frac{1}{n!} < \frac{1}{2^n}$$

The geometric series converges. It is

$$a + ar + ar^2 + ar^3 + \dots$$

with $a = r = \frac{1}{2}$

so the sum is $S = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$

So $\Sigma\, 1/n!$ converges too.

$\sum_{n=1}^{\infty} 1/n$ diverges

Harmonic series

The harmonic series, given by
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$
is divergent. A proof by term-by-term comparison is shown below.
Consider the harmonic series

$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} +$$

And another series, S_2 given

$$S_2 = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \dots$$

The terms in parentheses sum to $1/2$, so

$$S_2 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

which obviously diverges. We then notice that each and every term of $S1$ is greater or equal than the corresponding term in S_2

$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \dots$$

$$S_2 = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \dots$$

So, if S_2 diverges, S_1 has to diverge as well.

Integral test

Check figures 6.1 and 6.2 of the book. The sum

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

can be turned into an integral and evaluated:

$$\int_{n=1}^{\infty} \frac{1}{n} = \ln n \Big|_{n=1}^{\infty} = \ln n \Big|_{n=1}^{\infty} = \infty$$

first terms do not matter for convergence.

A decreasing series ($a_{n+1} \leq a_n$) will diverge if integral is infinite and converge if integral is finite.

The ratio test:

Let $\rho_n = \left|\frac{a_{n+1}}{a_n}\right|$; $\rho = \lim_{n \to \infty} \rho_n$

If $\begin{cases} \rho < 1 & \text{converges} \\ \rho = 1 & \text{use other test} \\ \rho > 1 & \text{diverges} \end{cases}$

Test $\sum \frac{1}{n!}$ $\quad \rho_n = \frac{n!}{(n+1)!} = \frac{\cancel{n!}}{(n+1)\cancel{n!}}$ $\quad \lim_{n \to \infty} \rho_n = \frac{1}{\infty} = 0$

Converges.

Alternating series

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - \dots$

Test: If an alternating is absolutely convergent, then it is convergent. That is, if

$\sum_n^{\infty} |a_n|$ converges, then $\sum_n^{\infty} (-1)^n a_n$ converges.

Ist, an alternating series may be convergent even though it is not absolutely convergent.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \quad \text{converges}$$

Leibniz test shows that one only needs to test if a_n goes to zero monotonically, that is, if

$$|a_{n+1}| \leq |a_n| \text{ and } \lim_{n\to\infty} a_n = 0$$

$$S_n = \sum_{k=0}^{n} (-1)^k a_k$$

$$S_m = \sum_{k=0}^{\infty} (-1)^k a_k$$

$$S_n - S_m = \sum_{k=0}^{n} (-1)^k a_k - \sum_{k=0}^{m} (-1)^k a_k$$

$$= \sum_{k=m+1}^{n} (-1)^k a_k$$

$$= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \dots - a_n$$

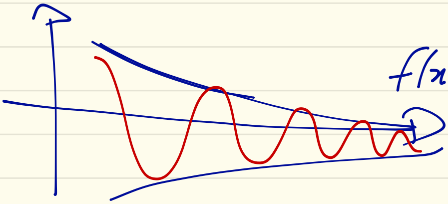
$$= a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \dots - a_n$$

$$\leq a_{m+1} \leq a_m$$

So, $S_m - S_n \leq a_m$

Since a_m converges to zero, the series converges

Example: damped oscillation.



$$f(x) = Ae^{-\lambda}e^{ix}$$

Convergence intervals of power series.

Not all Taylor series converge!

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{x^n}{n} + \dots$$

Converges only in the interval $-1 < x \leq 1$. Test the endpoints. For $x = -1$:

$$\log(1-1) = \log 0 = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = -\left(\Sigma \frac{1}{n}\right) = -\infty$$

For $x = 1$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$ converges,

but for $x > 1$, the ratio test shows divergence.

Test with ratio:

$$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots + (-1)^n \frac{x^n}{2^n}$$

Ratio: $\rho_N = \left| \frac{(-1)^{n+1} \cdot x^{n+1} / 2^{n+1}}{(-1)^n x^n / 2^n} \right| = \frac{\cancel{(-1)^n} (-1) \cdot \cancel{x^n} \cdot x}{\cancel{2^n} \cdot 2} \cdot \frac{\cancel{2^n}}{\cancel{x^n} \cancel{(-1)^n}}$

$$\rho_N = \left| \frac{x}{2} \right|$$

Converges for $-2 < x < 2$