Math Methods in Physics I

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Nov 8th, 2016

Class #21

surface sphere geodesic?

Sphere coordinates   Example airplane

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$d\vec{s} = dr\,\bar{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \qquad ds^2 = ds\cdot ds$$

On the sphere, $dr = 0$, use $r = 1$

$$ds^2 = d\theta^2 + \sin^2\theta\,d\phi^2$$

Length $\quad L = \int_A^B |ds| = \int_A^B \sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}$

$$d\theta^2 + \sin^2\theta\,d\phi^2 = d\theta^2\left(1 + \sin^2\theta\,\frac{d\phi^2}{d\theta^2}\right)$$
$$= d\theta^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$\theta = \theta(t)$; $\phi(t)$

$$ds = r\sqrt{\theta'^2 + \phi'^2\sin^2\theta}\,dt$$

$$S = \int_A^B \sqrt{\theta'^2 + \phi'^2\sin^2\theta}\,dt$$

$$ds^2 = d\theta^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$$L = \int_A^B |ds| = \int_A^B \sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

$$d\theta^2 + \sin^2\theta\, d\phi^2 = d\theta^2\left(1 + \sin^2\theta\, \frac{d\phi^2}{d\theta^2}\right)$$

$$= d\theta^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$$L = \int_A^B |d\theta|$$

$$s = \int |\theta'(t)|\, dt = |\theta(b) - \theta(a)|$$

Minimized if and only if $\phi' = 0$. Mus lie on a meridian of the sphere $\phi = \phi_0 \equiv$ const. In cartesian,

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{cte} \equiv \phi_0$$

$$\frac{y}{x} = \tan\phi_0 \qquad y\cos\phi_0 = x\sin\phi_0$$

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Path integral

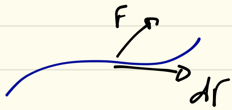
Stokes theorem, Gauss theorem?

Work

$$W = F \cdot d$$

infinitesimal

$$dW = F \cdot dr \quad ; \quad W = \int F \cdot dr$$



Given $F = xy\,\hat{i} - y^2\hat{j}$

Write the path in terms of a single variable:

Along a straight line. $y = 1/2\, x$; $dy = 1/2\, dx$. The limit for x is $(0,2)$:

$$W = \int_0^2 \left[x\left(\frac{x}{2}\right) dx - \left(\frac{1}{2}x\right)^2 \cdot \frac{1}{2} dx \right] = \int_0^2 \frac{3}{8} x^2 dx = \frac{x^3}{8}\Big|_0^2 = 1$$

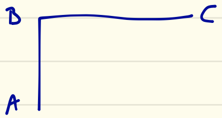
Along path 2, a parabola

$$y = \frac{1}{4}x^2 \quad ; \quad dy = \frac{1}{2} x\, dx$$

$$W_2 = \int_0^2 \left(x \cdot \frac{1}{4}x^2 dx - \frac{1}{16}x^4 \cdot \frac{1}{2} x\, dx \right) = \int_0^2 \left(\frac{1}{4}x^3 - \frac{1}{32}x^5 \right) dx =$$

$$= \frac{x^4}{16} - \frac{x^6}{192}\Big|_0^2 = \frac{2}{3}$$

Along path 3:



$$\int_A^B F\cdot dr + \int_B^C F\cdot dr$$

$$\int_{y=0}^{1} (0\cdot y\cdot 0 - y^2 dy) = -\frac{y^3}{3}\Big|_0^1 = -\frac{1}{3}$$

$$\int_{x=0}^{2} (x\cdot 1\cdot dx - 1\cdot 0) = \frac{x^2}{2}\Big|_0^2 = 2$$

Then the total $w_3 = -\frac{1}{3} + 2 = \frac{5}{3}$

Path 4:

Parameter t : $x = 2t^3$; $y = t^2$ $\therefore$ $dx = 6t^2 dt$; $dy = 2t\,dt$

$$\int_0^1 (2t^3\cdot t^2\cdot 6t^2 dt - t^4\cdot 2t\cdot dt) = \int_0^1 (12t^7 - 2t^5)dt = \frac{12}{8} - \frac{2}{6} = \frac{7}{6}$$

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Conservative Fields

Forces whose work do not depend on the path are called conservative

$$\int_A^B F\cdot dr = U(B) - U(A) = \int_A^B dU$$

$$dU = \nabla U\cdot dr = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

$$F = \nabla U \quad \therefore \quad \nabla\times F = \nabla\times\nabla U = 0$$

Conservative fields

Can be written in terms of a potential $\iff$ Have zero curl (irrotational)

Examples: Gravitational force
Electric field (electrostatics).

Gauss theorem and Stokes theorem

Green theorem

$$\int_a^b \frac{d}{dt} f(t)\,dt = f(b) - f(a)$$

Surface integral over A of $\frac{\partial P}{\partial y}$ is equal to line integral around C.

$$\iint \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b dx \int_{y_\ell}^{y_u} \frac{\partial P}{\partial y}\,dy = \int_a^b \left[P(x,y_u) - P(x,y_\ell)\right] dx$$

$$= -\int_a^b P(x,y_\ell)\,dx - \int_b^a P(x,y_u)\,dx$$

$$= -\oint P\,dx$$

$$\therefore \quad \oint_c P\,dx = -\iint_A \frac{\partial P}{\partial y}\,dx\,dy$$

If we did on x first

$$\iint \frac{\partial Q}{\partial x}\,dx\,dy = \oint_c Q\,dy$$

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_{\partial A} (P\,dx + Q\,dy) \qquad \partial A = C \ (\text{contour of } A)$$

So, we can say

$$W = \oint_C (F_x\,dx + F_y\,dy) = \iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy$$

$$W = \oint_C F \cdot dr = \iint_A (\nabla \times F) \cdot dA$$ Show the honeycomb pattern.

P and Q are arbitrary

Choose $Q = V_x$ and $P = -V_y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \nabla \cdot V$$

$dr = dx\,\hat{i} + dy\,\hat{j}$ (tangent)

Normal? n

$dr \cdot \hat{n} = 0$

$\hat{n} = a\,\hat{i} + b\,\hat{j}$

$a \cdot dx + b\,dy = 0$

$a = dy$; $b = -dx$ $\therefore$ $\hat{n} = \hat{i}\,dy - \hat{j}\,dx$

So,

$$P\,dx + Q\,dy = -V_y\,dx + V_x\,dy = (\hat{i}V_x + \hat{j}V_y)\cdot(\hat{i}\,dy - \hat{j}\,dx)$$

$$= V\cdot\hat{n}\,ds$$

So, 9.7:

$$\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \oint_{\partial A}(P\,dx + Q\,dy)$$

9.9: $$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \nabla\cdot V$$

$$\iint_A \nabla\cdot V\,dx\,dy = \oint_{\partial A} V\cdot\hat{n}\,ds$$

In 3D

$$\boxed{\iiint_V \nabla\cdot F\,dV = \oiint_A F\cdot\hat{n}\,dA}$$

For Stokes theorem, define

$$Q = V_y\ ;\ P = V_x\ ,\ \text{with}\ V = \hat{i}V_x + \hat{j}V_y \qquad (9.15)$$

So

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = (\nabla\times V)\cdot K$$

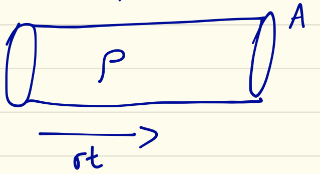
9.10 $(dr = \hat{i}dx + \hat{j}dy)$

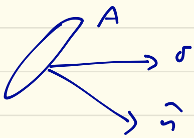
9.10 and 9.15 yield

$$P\,dx + Q\,dy = (\hat{i}V_x + \hat{j}V_y)\cdot(\hat{i}\,dx + \hat{j}\,dy) = V\cdot dr$$

$$\iint_A (\nabla\times F)\cdot K\, dA = \oint_{\partial A} F\,dr$$

Continuity equation



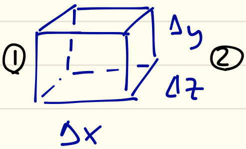


Amount of fluid crossing an area A' in unit time

$$\rho\cdot V = \rho\cdot v t A' = \rho v t A\cos\theta$$

Amount of water crossing a unit area in unit time is

$$\frac{\Delta m}{\Delta t\,\Delta A} = -\rho\vec{v}\cdot\hat{n} \qquad \rho\vec{v} = \vec{p}$$



Mass entering through 1:

$$\left.\frac{\Delta m}{\Delta t\,\Delta A}\right|_1 = -\rho\vec{v}\cdot\hat{n} \qquad \frac{\Delta m}{\Delta t} = -\rho\vec{v}_1\cdot\hat{n}\,\Delta A_1$$

Mass entering through 2:

$$\frac{\Delta m}{\Delta t} = -\rho \vec{v}_2 \cdot \hat{n} \Delta A_2$$

Net flow:

$$\left(\frac{\Delta m}{\Delta t}\right)\Big|_2 + \left(\frac{\Delta m}{\Delta t}\right)_1 = \vec{p}_1 \cdot \hat{n} \Delta A_1 + \vec{p}_2 \cdot \hat{n} \Delta A_2$$

$$\Delta A_1 = \Delta A_2 = \Delta y \Delta z$$

$$\left(\frac{\Delta m}{\Delta t}\right)\Big|_2 + \left(\frac{\Delta m}{\Delta t}\right)_1 = \vec{p}_1 \cdot \hat{n} \Delta A_1 + \vec{p}_2 \cdot \hat{n} \Delta A_2$$

$$= \left(\vec{p}_2 \cdot \hat{n}_2 + p_1 \cdot \hat{n}_1\right) \Delta y \Delta z$$

$$\vec{p}_2 \cdot \vec{n}_2 = \vec{p}_2 \cdot (\hat{x}) = p_{x_2} \quad \Rightarrow \quad \vec{p}_1 \cdot \hat{n}_1 = p_1 (-\hat{x}) = -p_{x_1}$$

$$\left(\frac{\Delta m}{\Delta t}\right)\Big|_2 + \left(\frac{\Delta m}{\Delta t}\right)_1 = (p_{x2} - p_{x1}) \Delta y \cdot \Delta z = \Delta p_x \Delta y \Delta z$$

Take limits now $\lim_{\Delta x \to 0} \Delta p_x = \lim_{\Delta x \to 0} \frac{\Delta p_x}{\Delta x} \Delta x = \frac{\partial p_x}{\partial x} dx$

$$\frac{\partial m}{\partial t}\Big|_2 + \frac{\partial m}{\partial t}\Big|_1 = \frac{\partial p_x}{\partial x} dx\, dy\, dz$$

For the other faces of the cube:

$$\frac{\partial m}{\partial t} = \left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z}\right) dx\,dy\,dz = \bar{\nabla}\cdot p \; dV$$

$$\frac{\partial m}{\partial t\,\partial V} = \nabla\cdot\vec{p}$$

Net outflow per unit volume evaluated at a point.

$$p = \rho\cdot\vec{v} \qquad m\,dV = d\rho$$

$$\boxed{\frac{\partial \rho}{\partial t} = \nabla\cdot(\rho\vec{v})}$$

$$\frac{\partial \rho}{\partial t} = \rho\,\bar{\nabla}\cdot v + (v\cdot\nabla)\rho$$

$$\boxed{\frac{\partial \rho}{\partial t} + (v\cdot\nabla)\rho = -\rho\,\bar{\nabla}\cdot v}$$

Mass of fluid going outside

$$\iiint \frac{\partial m}{\partial t}\,dV = \oiint m\,v\cdot n\,dA$$

Order of magnitude estimate.

$$\iiint \frac{\partial \rho}{\partial t}\,dV = \oiint \rho v\cdot n\,dA$$

$$\iiint \nabla(\rho v)\,dV = \oiint \rho v\cdot n\,dA$$

Gauss theorem / Divergence theorem

$$\boxed{\iiint \nabla\cdot F \, dV = \oiint F\cdot n \, dA}$$

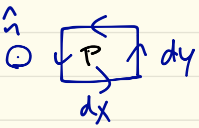
Give as example what I did in a paper

$$t_r = \frac{E}{\dot{E}} = \frac{E}{\nabla\cdot F} = \frac{\int E \, dV}{\int \nabla\cdot F \, dV} = \frac{E\cdot V}{F\cdot A} = \frac{(C_v \rho T)\cdot \frac{4\pi}{3} H^3}{\sigma T^4 \, 4\pi H^2} \sim \frac{C_v \rho H}{3\sigma T^3}$$

Stokes theorem

Circulation: line integral of vector field

$$\oint F \, dr$$



$$\oint F\cdot dr = \iint (\nabla\times F)\cdot \hat{k} \, dx \, dy = \iint (\nabla\times F)\cdot \hat{n} \, dA$$

Back to green's theorem; $\Leftarrow$

Main application of these theorems is in electrodynamics

$$\oint_C B \, dr = \mu_0 \iint J\cdot n \, dA$$

Biot-Savart law
to Ampère law

$$\oiint (\nabla\times B)\cdot n \, dA \rightarrow \nabla\times B = \mu_0 J$$