Math Methods in Physics I

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Class #26

Parseval Theorem

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$$

Square and average over $-\pi, \pi$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx$$

Square of sine or co-sine over period is $1/2$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (a_0/2)^2 dx = (a_0/2)^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (a_n \cos nx)^2 dx = \frac{a_n^2}{2\pi}\int_{-\pi}^{\pi} (\cos nx)^2 dx = \frac{a_n^2}{2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (b_n \sin nx)^2 dx = \frac{b_n^2}{2\pi}\int_{-\pi}^{\pi} (\sin nx) dx = \frac{b_n^2}{2}$$

The cross terms with $\sin nx \sin mx$ or $\cos nx \cos mx$ or $\sin nx \cos mx$ all zero

So:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2$$

Unchanged if period is different

$$\frac{1}{2L}\int_{-L}^{L} [f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2$$

For the complex series

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2dx = \sum_{-\infty}^{\infty}|c_n|^2$$

The square of the amplitude is the sum of the square of the amplitudes. How does that play out?

For a light wave, following Maxwell equations

$$\nabla\cdot E = 0 \qquad \nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla\times B = \frac{1}{c}\frac{\partial E}{\partial t}$$

Taking the curl of Faraday law

$$\nabla\times\nabla\times E = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times B \quad \Rightarrow \quad \cancel{\nabla(\nabla\cdot E)} - \nabla^2 E = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \qquad \nabla^2 B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t}$$

Consider wave solutions of the form

$$E = \hat{a}_1 E_0 e^{i(K\cdot r - \omega t)}$$
$$B = \hat{a}_2 B_0 e^{i(Kr - \omega t)}$$

The energy in an electric field is E^2. So, if we expand a light wave in co-sine series

$$E(x,t) = \sum_{n=0}^{\infty} E_n \cos(K_n x - \omega_n t)$$

Parseval theorem states that

$$E^2 = \sum_{n=-\infty}^{\infty} |E_n|^2$$

i.e. the energy of the signal is the sum of the energy of the individual components.

Odd / even transforms

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-iKx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(Kx)\, dx - i\int_{-\infty}^{\infty} f(x)\sin(Kx)\, dx$$

For odd functions, $f(x)\cos(Kx)$ is odd and the integral cancels

$$g(k) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} f(x)\sin(Kx)\, dx = -\frac{i}{\pi}\int_{0}^{\infty} f(x)\sin(Kx)\, dx$$

$$f(x) = \int_{-\infty}^{\infty} g(K)\, e^{iKx}\, dK = 2i\int_{0}^{\infty} g(K)\sin(Kx)\, dk$$

Substitute $g(K)$ to remove the imaginary factor

$$f(x) = 2i\left(\frac{-i}{\pi}\right)\int_0^{\infty} \sin Kx\, dK \int_0^{\infty} f(x)\sin Kx\, dx$$

$$= \frac{2}{\pi}\int_0^{\infty} g(K)\sin Kx\, dK$$

$$g(K) = \int_0^{\infty} f(x)\sin Kx\, dx$$ for odd functions

For even functions, keep the co-sine

$$f(x) = \frac{2}{\pi}\int_0^\infty g(k)\cos kx \, dk$$

$$g(k) = \int_0^\infty f(x)\cos kx \, dx$$ for even functions

A note on coefficients. We derive the relation between the Fourier transform and the inverse by the integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{ik(x-u)}\, du\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} f(u) e^{-iku} du$$

Then we defined the fourier pair

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad \text{and} \quad f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk$$

The constrain is that when multiplied the coefficients must multiply to 2π. We are free to choose how. So, the fourier pair can be

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk$$

$$g(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)\, e^{ikx} dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)\, e^{ikx} dk.$$

The three are used in the literature.

Application to Poisson equation to the gravitational potential

$$g = -\frac{GM}{r^2} \qquad \oint g \cdot dA = -GM\int \frac{1}{r^2}\, dA = -4\pi GM$$

$$M = \int \rho\, dV \qquad \therefore \oint g\, dA = -4\pi G\int \rho\, dV \qquad \int \nabla\cdot g\, dV = -4\pi G\int \rho\, dV$$

$$\nabla\cdot g = -4\pi G\rho \qquad \text{use } g = -\nabla\phi$$

$$\nabla^2\phi = 4\pi G\rho \qquad \text{set } 4\pi G = 1$$

$$\nabla^2\phi = \rho$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \rho$$

Fourier transform $\phi = \sum_n \phi_n e^{iK_n x}$ $\therefore \frac{\partial}{\partial x}\phi = i\sum_n K_n \phi_n e^{iK_n x}$

$$\frac{\partial^2\phi}{\partial x^2} = -\sum_n K_n^2 \phi_n e^{iK_n x}$$

$$\sum_n \left(\frac{\partial^2}{\partial x^2}\hat\phi + \frac{\partial^2}{\partial y^2}\hat\phi + \frac{\partial^2}{\partial z^2}\hat\phi\right) = \sum_n \hat\rho$$

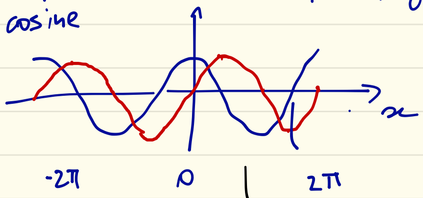
$$\sum_n (-K_x^2 - K_y^2 - K_z^2)\hat\phi = \sum_n \hat\rho$$

$$-K^2\hat\phi = \hat\rho \quad \therefore \quad \hat\phi = -\frac{\hat\rho}{K^2} \quad \therefore \quad \boxed{\phi = -F^{-1}\left(\frac{\hat\rho}{K^2}\right)}$$

Extra credit homework, compute the wavectors?

k: wavevector, defines the direction of propagation

See: cosine



$f(x) = \cos(x)$

$f(x) = \cos(x - \pi/2)$

$2\pi \to \frac{3\pi}{2}$

$\frac{3\pi}{2} \to \pi$

$\nabla \cdot E = 0 \Rightarrow k \cdot \hat{a}_1 = 0$; $\nabla \cdot B = 0 \Rightarrow k \cdot \hat{a}_2 = 0$

$k \cdot E = 0$ $\qquad$ $k \cdot B = 0$

Both the electric and magnetic fields are perpendicular to the direction of motion: the wave is transverse. What do we get from Faraday and Ampère laws?

$$\left.\begin{aligned} i k \times \hat{a}_1 E_0 &= \frac{1}{c} i \omega \hat{a}_2 B_0 \\ i k \times \hat{a}_2 B_0 &= -\frac{1}{c} i \omega \hat{a}_1 E_0 \end{aligned}\right\} \qquad \begin{aligned} k \times E &= \frac{\omega}{c} B \\ k \times B &= -\frac{\omega}{c} E \end{aligned}$$

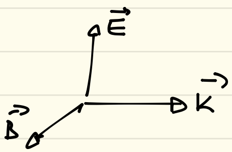
$$k \times E = \frac{\omega}{c} B \qquad\qquad k \times B = -\frac{\omega}{c} E$$

Dot this with E

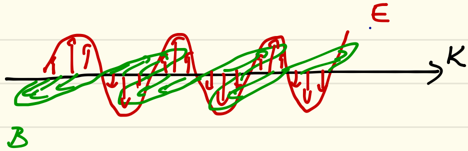
$$E \cdot (k \times E) = \frac{i\omega}{c} E \cdot B$$

$$k \cdot \cancel{(E \times E)} = \frac{i\omega}{c} E \cdot B \quad \therefore \quad E \cdot B = 0$$

E and B are perpendicular!



$$E_0 = \frac{\omega}{kc} B_0 \qquad B_0 = \frac{\omega}{kc} E_0$$



The amplitudes are in phase!

Recall that E and B are complex. If the relationship was $E \propto i \cdot B$, the fields would be out of phase.

Dispersion: $\omega^2 = c^2 k^2$

$$\therefore \ \omega = ck \quad \Rightarrow \quad \boxed{E_0 = B_0}$$

Parseval theorem for Fourier integrals

$$\bar{g}_1(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x) e^{ikx} dx$$

Multiply by $g_2(k)$ and integrate in k

$$\int_{-\infty}^{\infty} \bar{g}_1(k) g_2(k)\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \bar{f}_1(x) e^{ikx} dx\right] g_2(k)\, dk$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x)\, dx \left[\int_{-\infty}^{\infty} g_2(k) e^{ikx} dk\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x) f_2(x)\, dx$$

$$\int_{-\infty}^{\infty} \bar{g}_1(k)\, g_2(k)\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x) f_2(x)\, dx$$

Set $f_1 = f_2 = f$ and $g_1 = g_2 = g$

$$\int_{-\infty}^{\infty} |g(k)|^2 dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2 dx$$

Parseval theorem

Think of a vector V. As seen in coordinate system S with basis vector $\hat{e}_i$, it can be written

$$V = \sum_i V_i \hat{e}_i$$

where V_i are the components of V in S. As seen from another coordinate system S' with basis vectors $\hat{e}'_i$, it has a representation

$$V = \sum_i V'_i \hat{e}'_i$$

obviously the length of the vector is independent of the coordinate system used to represent it. In other words, we must have

$$\sum_i V_i^2 = \sum_i (V'_i)^2$$

Proceeding with the analogy, for a function $f(x)$ one can have a position space representation in δ-function basis as

$$f(x) = \int f'(x)\, \delta(x - x')\, dx$$

where the "component" of $f(x)$ along the "basis vector" $\delta(x-x')$ is $f(x')$ and we sum over all the possible axes. One can look at the same function in Fourier space representation as

$$f(x) = \int g(k)\, e^{-ikx}\, dk$$

where e^{-ikx} are the "basis vectors" and $g(k)$ are the components of $f(x)$ along these basis vectors. You would then agree that

$$\int |f(x)|^2 dx = \int |g(k)|^2 dk$$

So, Parceval theorem is just the restatement of the invariance of the length of a vector, independent of the representation used.

In our case it means that the energy in real space is equal to the energy in Fourier space.

Image processing example

Convolution theorem

Linearization

Next class.