# Math Methods in Physics I

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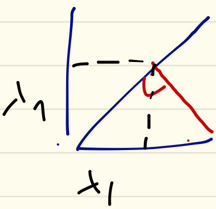
class #16

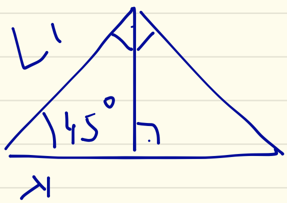
# Projections

10/20/16

Example: projection onto line $a=(1,1)$. Try it through trigonometry



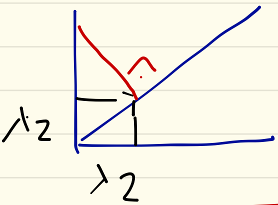

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{12} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

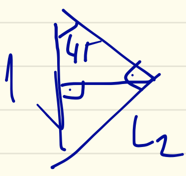



$$L_1 = \cos 45^\circ = 1/\sqrt{2}$$

$$\lambda_1 = L_1 \cdot \cos 45 = 1/2$$

Same for $(0,1)$: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$



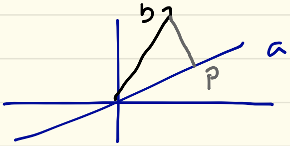




$$L_2 = \cos 45^\circ = 1/\sqrt{2}$$

$$\lambda_2 = L_2 \cos 45 = 1/2$$

$$\boxed{A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

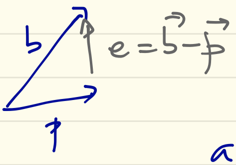
# Projections: The Linear Algebra way




The closest point P is at the intersection formed by a line through b that is orthogonal to a.

If $p$ lies in the line defined by a, then

$$p = \lambda a$$




We also know that a and e are perpendicular

$$a \cdot e = 0$$

$$a^T (b - p) = 0$$

$$a^T (b - \lambda a) = 0$$

$$\lambda a^T a = a^T b \quad \therefore \quad \lambda = \frac{a^T b}{a^T a}$$

$$\text{So, } p = a\lambda = a \left( \frac{a^T a}{a^T b} \right)$$

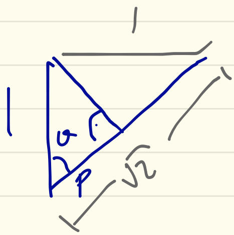
Example: projection of $b = (1,0)$ on line $a = (1,1)$

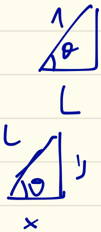
$$a^T a = a \cdot a = (1 \; 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

$$a^T b = (1 \; 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$x = 1/2$$

$p = a\lambda = a/2$





$L = \cos\theta = \frac{1}{\sqrt{2}}$

$x = L\cos\theta = \frac{1}{2}$

We'd like to write it in terms of a projection matrix $P$

$p = Pb$ The matrix $P$ that transforms the vector $b$ into $p$

$p = Pb$

$= a\frac{a^Tb}{a^Ta}$ $p = P\cdot b = a\frac{a\cdot b}{a\cdot a} = P\cdot b$ ; $P = a\left(\frac{a}{a\cdot a}\right)$

$p_i = P_{ij}b_j = \frac{a_i a_j b_j}{a_k a_k}$ $P_{ij} = \frac{a_i a_j}{a_k a_k}$

what is $a_i a_j$? Since $a_k a_k$, the inner product, is a scalar, $a_i a_j$, having two indices, has to be a matrix. This is the outer product.

# Outer product

$$w_{ij} = u_i v_j$$

$$s = a \cdot b = a^T b \qquad (\text{scalar})$$
$$w = a \otimes b = a b^T \qquad (\text{matrix})$$

a and b are vectors, or $n \times 1$ matrices
So, $a^T b$ is $(1 \times n)(n \times 1)$, so a scalar

$ab^T$ is a $(n \times 1)(1 \times m)$, so a $n \times m$ matrix

For instance, if $n = 4$ and $m = 3$

$$a \otimes b = a b^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} (b_1 \ b_2 \ b_3) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \\ a_4 b_1 & a_4 b_2 & a_4 b_3 \end{pmatrix}$$

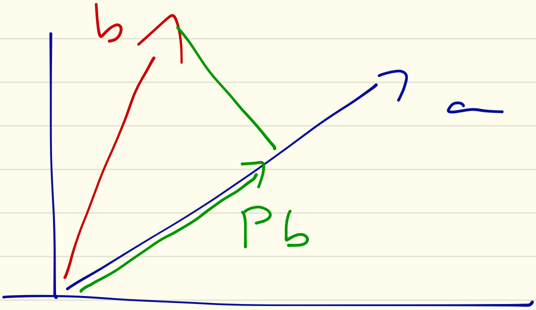
So, the projection matrix is

$$\boxed{P_{ij} = \frac{a_i a_j}{a_k a_k} \ ; \quad P = \frac{a \otimes a}{a \cdot a} \ ; \quad P = \frac{a a^T}{a^T a}}$$

Example. $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad a a^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \ 1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$a^T a = (1 \ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

$P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ This matrix projects any vector in the line defined by $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.




In the exam, you were asked the matrix that projects onto $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$P = \frac{a a^T}{a^T a}$ ; $a^T a = (2\ 1\ 3)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 14$

$a a^T = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}(2\ 1\ 3) = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{pmatrix}$

$P = \frac{1}{14}\begin{pmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{pmatrix}$

Properties of P

P has rank 1.

P is symmetric

$P^2 b = Pb$

The projection of a vector already on the line through $a$ is just that vector.

$P$ operating on any vector on the line $a$ is just that vector. So, operating twice won't change it.

In general

$$P^T = P \qquad P^2 = P$$

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

square it

$$P^2 = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

## Projection on higher dimensions

Project a vector $b$ onto closest point $p$ in a plane

$$p = Pb$$

Basis of plane:

Say we are in Cartesian space. The unit vectors are $\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We can write any vector in that plane as a linear combination of $\hat{x}$ and $\hat{y}$:

$$\vec{p} = \lambda_1 \hat{x} + \lambda_2 \hat{y} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In matrix form, we can write

$$\vec{p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 & \hat{y}_1 \\ \hat{x}_2 & \hat{y}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = A\vec{\lambda}$$

; A in this case the identity matrix

In general, we can assume any basis vectors, as long as they are orthogonal

$$\hat{x} \to \hat{a}_1 \qquad \hat{y} \to \hat{a}_2$$

vectors will be written $\lambda_1 \hat{a}_1 + \lambda_2 \hat{a}_2$

$$\vec{p} = \lambda_1 \hat{a}_1 + \lambda_2 \hat{a}_2 = A\vec{\lambda}$$

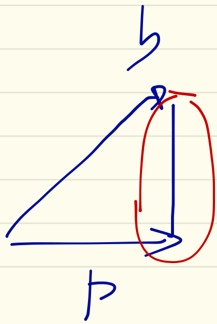
with the columns of A being $\hat{a}_1$ and $\hat{a}_2$.

$$A = \begin{pmatrix} (a_1)_1 & (a_2)_1 \\ (a_1)_2 & (a_2)_2 \end{pmatrix}$$

So, the projection Pb we can write as $A\vec{\lambda}$, where A is the matrix whose columns are the basis vectors of the plane.

$$p = \lambda_1 \hat{a}_1 + \lambda_2 \hat{a}_2 = A\vec{\lambda}$$

$$e = b - p$$




perpendicular to both $\hat{a}_1$ and $\hat{a}_2$

$$b - p = b - A\vec{\lambda}$$

$$a_1 \cdot (b-p) = a_2 (b-p) = 0$$

$$a_1^T (b - A\vec{\lambda}) = 0 \quad \text{and} \quad a_2^T (b - A\lambda) = 0$$

In matrix form

$$A^T (b - A\vec{\lambda}) = 0$$

$$A^T b - A^T A \vec{\lambda} = 0$$

$$A^T A \vec{\lambda} = A^T b$$

when projecting on a line $A^T A$ was a number. Now it is a square matrix. So, instead of dividing by $a^T a$, call $M = A^T A$

$$M\vec{\lambda} = A^T b$$

we need to multiply by $M^{-1}$

$$M^{-1} M \vec{\lambda} = M^{-1} A^T b \qquad \vec{\lambda} = M^{-1} A^T b$$

and $M^{-1} = (A^T A)^{-1}$

So $\vec{\lambda} = (A^T A)^{-1} A^T b$

$$p = Pb = A\vec{\lambda} = A (A^T A)^{-1} A^T b$$

$$\therefore \boxed{P = A (A^T A)^{-1} A^T}$$

Example:

Find the matrix that projects vectors onto the plane defined by the vectors $a_1=(1,2,1)$ and $a_2=(1,-1,1)$.

Are they perpendicular? $1-2+1=0$ yes,

Any vector can be written as $\vec{p}=\lambda_1 a_1+\lambda_2 a_2 = A\lambda$ with $A$ being

$$A=\begin{bmatrix}1 & 1\\ 2 & -1\\ 1 & 1\end{bmatrix}$$ A is not square!

The projection matrix onto this plane is

$$\boxed{P = A(A^TA)^{-1}A^T}$$

$$A=\begin{bmatrix}1 & 1\\ 2 & -1\\ 1 & 1\end{bmatrix} \quad A^T=\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & 1\end{bmatrix} \quad A^TA=\begin{bmatrix}6 & 0\\ 0 & 3\end{bmatrix}=3\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$$

$$(A^TA)^{-1}=\begin{bmatrix}1/6 & 0\\ 0 & 1/3\end{bmatrix}$$

$$\therefore A(A^TA)^{-1}A^T=\begin{bmatrix}1 & 1\\ 2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1/6 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1/6 & 1/3\\ 1/3 & -1/3\\ 1/6 & 1/3\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & 1\end{bmatrix}=\begin{bmatrix}1/2 & 0 & 1/2\\ 0 & 1 & 0\\ 1/2 & 0 & 1/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}$$

Apply to $\begin{pmatrix}1\\8\\1\end{pmatrix} \rightarrow \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}$ (show plotter)

# Application: Least squares

Collection of data $(t, b)$

$$\{(1,1)\ (2,2)\ (3,2)\}$$

Find the closest line $b = C + Dt$ to that collection.
If it went through all three points, we'd have

$$C + D = 1$$
$$C + 2D = 2$$
$$C + 3D = 2$$

Which is equivalent to

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{\lambda}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

But the line does not go through all three points. Instead, let us look for the line whose projections of $(1,1)$ $(2,2)$, $(3,2)$ fall in it

$$A^T A \vec{\lambda} = A^T b$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \rightarrow \begin{array}{l} 3C + 6D = 5 \\ 6C + 14D = 11 \end{array}$$

$$-\cancel{6C} - 12D = -10$$
$$\cancel{6C} + 14D = 11 \qquad D = 1/2 \qquad 3C = 2 \;;\; C = 2/3$$

Least squares fit of the points $(t, b) = \{(1,1)\ (2,2)\ (3,2)\}$

$$\boxed{b = 2/3 + t/2}$$