Math Methods in Physics I

Prof Wladimir Lyra

Dec 6^{th}, 2016

Class #28

Orbital motion

$$\ddot{r} - r\dot{\phi}^2 = -\Omega^2 r \qquad \dot{r} = v_r$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \qquad \dot{\phi} = \frac{v_\phi}{r}$$

$$\frac{dv_r}{dt} - r\frac{v_\phi^2}{r^2} = -\Omega^2 r$$

$$r\frac{d}{dt}\left(\frac{v_\phi}{r}\right) + 2v_r\frac{v_\phi}{r} = 0 \Rightarrow \frac{dv_\phi}{dt} - \frac{v_\phi r}{r^2}\frac{dr}{dt} + 2\frac{v_r v_\phi}{r} = 0$$

$$\begin{cases} \dfrac{\partial v_r}{\partial t} - \dfrac{v_\phi^2}{r} = -\Omega^2 r \\ \dfrac{\partial v_\phi}{\partial t} + v_r\dfrac{\partial}{\partial r}v_\phi + \dfrac{v_r v_\phi}{r} = 0 \end{cases}$$

$$u_\phi = u_\phi' + \Omega r$$

$$\partial_t u_r' - 2\Omega v_\phi' = 0 \qquad u_r' = \delta u_r\, e^{i(kr - \omega t)}$$

$$\partial_t v_\phi' + 2\Omega v_r' = 0 \qquad u_\phi' = \delta u_\phi\, e^{i(kr - \omega t)}$$

$$-i\omega u_r' - 2\Omega u_\phi' = 0$$

$$-i\omega u_\phi' + 2\Omega u_r' = 0$$

Solve: $\omega^2 = 4\Omega^2 = \kappa^2$ $\boxed{\kappa = 2\Omega}$

Hammer that with Fourier decomposition

$$\frac{\partial}{\partial t} \to -i\omega$$

$$\nabla \to ik$$

So differential equations become algebraic equations.

$$F(f(x)) = \int f(x)\, e^{-ikx}\, dx$$

$$F\left(\frac{df}{dx}\right) = \int \frac{df}{dx}\, e^{-ikx}\, dx$$

$$u = e^{-ikx} \quad ; \quad dv = \frac{df}{dx}\, dx$$

$$du = -ik\, e^{-ikx} dx \quad ; \quad v = f(x)$$

$$\therefore F(f') = uv - \int v\, du = e^{-ikx} f(x) \Big|_{-\infty}^{\infty} - \int f(x)(-ik)\, e^{-ikx}\, dx$$

$$= ik \int f(x)\, e^{-ikx}\, dx$$

$$F\left(\frac{df(x)}{dx}\right)(k) = ik\, \hat{f} \qquad \frac{d\hat{f}}{dx} = ik\, \hat{f}$$

other way to show: inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk \quad ; \quad g(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \hat{f}$$

$$\frac{d}{dx} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)\, ik\, e^{ikx}\, dk$$

$$F\left(\frac{df}{dx}\right) = g(k)\, ik = ik\, \hat{f}$$

— // —

Back to wave equation

$$\frac{\partial^2}{\partial t^2} f - v^2 \nabla^2 f = 0$$

$$-\omega^2 + v^2 k^2 = 0 \qquad \boxed{\omega^2 = v^2 k^2} \quad \text{(solution)}$$

$$e^{i(kx - \omega t)} \qquad \omega = vk \qquad e^{i(kx - vkt)} = e^{ik(x - vt)} = \cos\left[k(x - vt)\right]$$

Harmonic Oscillator

$$f = -Ax \qquad m\ddot{x} = -Ax \quad -m\omega^2 \hat{x} = -A\hat{x} \qquad \omega = \sqrt{\frac{A}{m}} \qquad x = \cos\sqrt{\frac{A}{m}}\, t$$

Poisson equation

$$\nabla^2 V = \rho \qquad -k^2 \hat{V} = \hat{\rho} \qquad \hat{V} = -\frac{\hat{\rho}}{k^2} \qquad V = -F^{-1}\left(\frac{\hat{\rho}}{k^2}\right)$$

Convolution theorem

$$\int_0^t g(t-\tau)\,h(\tau)\,d\tau = g * h$$

Multiply two fourier transforms

$$g_1(\alpha)\,g_2(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(v)\,e^{-i\alpha v}\,dv \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(u)\,e^{-i\alpha u}\,du$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\alpha(v+u)}\,f_1(v)\,f_2(u)\,dv\,du$$

$x = u + v \qquad dx = dv$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\alpha x}\,f_1(x-u)\,f_2(u)\,dx\,du$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{-i\alpha x}\left[\int_{-\infty}^{\infty} f_1(x-u)\,f_2(u)\,du\right]dx$$

Define convolution $f_1 * f_2 = \int_{-\infty}^{\infty} f_1(x-u)\,f_2(u)\,du$

$$g_1 \cdot g_2 = \frac{1}{2\pi}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f_1 * f_2 \; e^{-i\alpha x}\,dx\right] = \frac{1}{2\pi}F(f_1 * f_2)$$

So $g_1 g_2 = \frac{1}{2\pi}F(f_1 * f_2)$

$f_1 f_2 = F^{-1}(g_1 * g_2)$