Math Methods in Physics I

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Class #1

Proofs of Euler's formula and Euler's identity

Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler's identity ($\theta = \pi$)

$$\boxed{e^{i\pi} + 1 = 0}$$

Let's start with the concept of Power Series. Let us conjecture the existence of the following identity

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n x^n}$$

and look for the coefficients a_n that make it true. A way to do so is to compute the derivatives of $f(x)$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_n x^n + \dots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots + n \cdot a_n x^{(n-1)} + \dots$$

$$f''(x) = 2a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \dots + n(n-1) a_n x^{(n-2)} + \dots$$

$$f'''(x) = 3!\, a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x + \dots + n(n-1)(n-2) a_2 x^n + \dots$$

$$f^{(4)}(x) \quad 4!\, a_4 + \dots + n(n-1)(n-2)(n-3) x^{n-4} + \dots$$

$$f^{(n)}(x) = n!\, a_n + \dots + \text{terms with } x$$

Evaluate the series and the derivatives for $x = 0$

$f(0) = a_0$; $f'(0) = a_1$; $f''(0) = 2a_2$; $f'''(0) = 3!\, a_3$

$f^{(n)}(0) = n!\, a_n$

That is : $\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$

So,

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0)}$$

$$\boxed{f(x) = \sum_{x=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

The expression we found can be generalized to

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \ldots$$

with

$$\boxed{a_n = \frac{f^{(n)}(a)}{n!}}$$

$$\boxed{f(x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \ldots + \frac{x^n}{n!} f^{(n)}(a)}$$

$$\boxed{f(x) = \sum_{x=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n}$$

Application:

$$e^x = e^0 + x e^0 + \frac{x^2}{2!} e^0 + \frac{x^3}{3!} e^0 + \ldots + \frac{x^n}{n!} e^0 = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$\sin(x) = \sin(0) + x \frac{d}{dx} \sin(x)\Big|_0 + \frac{x^2}{2!} \frac{d^2}{dx^2} \sin(x)\Big|_0$$

$$+\frac{x^3}{3!}\frac{d^3}{dx^3}\sin(x)\Big|_0 + \dots + \frac{x^n}{n!}\frac{d^n}{dx^n}\sin(x)\Big|_0 + \dots$$

$$\sin x = 0 + x\cos(0) + \frac{x^2}{2}(-\sin(0)) + \frac{x^3}{3!}(-\cos(0)) + \dots$$

$$+\frac{x^4}{3!}\sin(0) + \frac{x^5}{5!}\cos(0) + \frac{x^6}{6!}(-\sin(0)) + \frac{x^7}{7!}(-\cos(+\dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\boxed{\sin(x) = \sum_{n=0} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}}$$

$$\cos(x) = \cos 0 + \frac{d\cos x}{dx}\Big|_0 \cdot x + \frac{d^2\cos x}{dx}\Big|_0 \frac{x^2}{2} + \frac{d^3\cos x}{dx^3}\frac{x^3}{3!} + \dots$$

$$= 1 - \cancel{\sin(0)\,x} - \cos(0)\frac{x^2}{2} + \cancel{\sin(0)}\frac{x^3}{3!} + \cos(0)\frac{x^4}{4!}$$

$$-\cancel{\sin(0)\frac{x^5}{5!}} - \cos 0\frac{x^6}{6!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}}$$

So, now let's prove Euler's formula. If

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Then

$$e^{ix} = \sum_{n=0}^{\infty} i^n \frac{x^n}{n!}$$

Separate into even and odd powers:

$$e^{ix} = \sum_{n=0}^{\infty} \left(i^{2n} \frac{x^{2n}}{2n!} \right) + \left(i^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \right)$$

Notice that

$$i^{2n} = (i^2)^n = (-1)^n$$

And that

$$i^{2n+1} = i \cdot i^{2n} = i(-1)^n$$

So,

$$e^{ix} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} + i(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \right) + i \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

And these are the series of sin x and cos x. Thus,

$$\boxed{e^{ix} = \cos x + i \sin\theta}$$

Q. E. D.

Taking the special case $x = \pi$:

$$e^{i\pi} = \cos\pi + \cancel{i\sin\pi} = -1$$

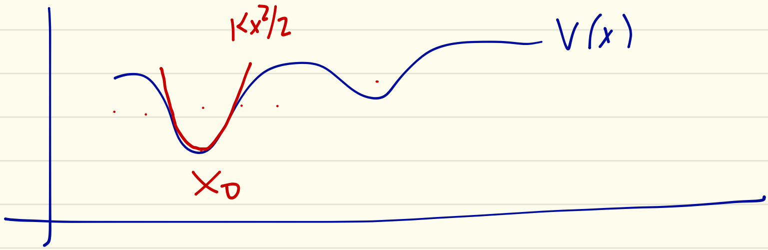
$$\therefore \boxed{e^{i\pi} + 1 = 0}$$

Q.E.D

Taylor series application in physics:

Harmonic oscillator: $F = -Kx$; $V = Kx^2/2$

Approximate a potential well around the energy minimum x_0



$$V(x) = V(x_0) + \cancel{V'(x_0)(x-x_0)} + V''(x_0)\frac{(x-x_0)^2}{2!} + \dots$$

$V(x_0)$ is a minimum, so $V'(x_0)$ drops

$$V(x) = V(x_0) + V''(x_0)\frac{(x-x_0)^2}{2} + O(x^3)$$

where $O(x^3)$ stands for terms of 3rd order and higher.

$V(x_0)$ is arbitrary. Coordinate transformation

$$\boxed{V(x) \sim V''(0)\frac{x^2}{2} = \frac{Kx^2}{2}}$$

Another neat trick. The small number approximation:

$$y(x) = (1+x)^n \approx 1 + nx \quad \text{for } x \ll 1$$

Proof:

$$y(x) = y(0) + y'(0)\frac{x}{2} + y''(0)\frac{x^2}{2} = 1 + n(1+x)^{n-1}\Big/_{x=0} x + n(n-1)(1-x)^{n-2}\Big/_{x=0}\frac{x^2}{2}$$

$$= 1 + nx + n(n-1)\frac{x^2}{2}$$

For $x \ll 1$, to 1st order

$$\boxed{(1+x)^n \simeq 1 + nx}$$

Try some examples:

$$(1+10^{-6})^2 = 1.000002000001$$
$$(1+2\cdot 10^{-6}) = 1.000002$$

$$(1+10^{-6})^3 = 1.000003000002998$$
$$(1+3\cdot 10^{-6}) = 1.000003$$

Adding the 2nd order term has little effect:

$$1 + 3\cdot 10^{-6} + 6\cdot 10^{-12} = 1.000003000006$$