Math Methods in Physics I

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class #10

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

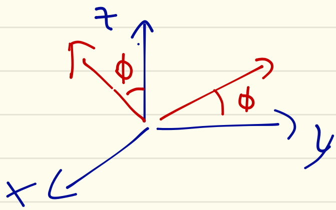
$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\hat{x} + \cos\theta\,\dot{\theta}\,\hat{y} = \dot{\theta}\,\hat{\theta}$$
$$\dot{\hat{\theta}} = -\cos\theta\,\dot{\theta}\,\hat{x} - \sin\theta\,\dot{\theta}\,\hat{y} = -\dot{\theta}\,\hat{r}$$

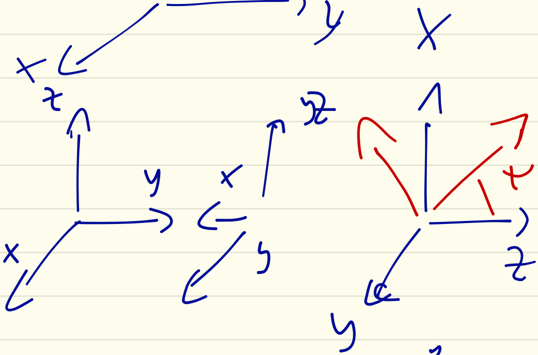
$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \dot{\theta} \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

$$\hat{r}_i = R_{ij}\,\hat{x}_j \qquad \dot{\hat{r}}_i = \dot{R}_{ij}\,\hat{x}_j$$

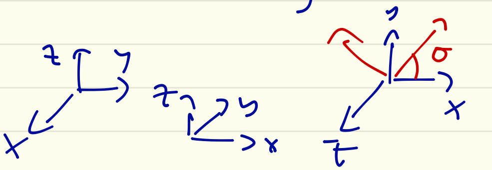
x-axis
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$



y-axis
$$\begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$



z axis
$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Functions of Matrices

A can I take e^A?

$$A = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$A^3 = -A \ ; \ A^4 = I$$

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots$$

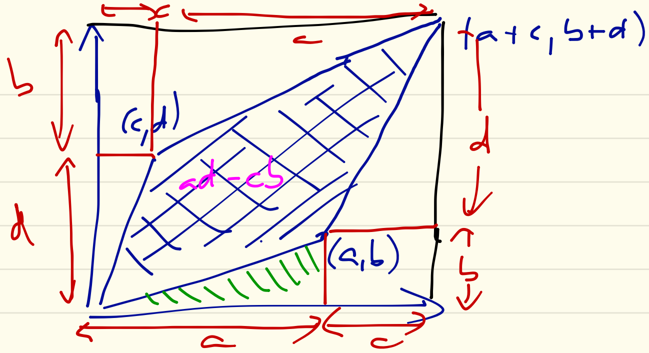
So

$$e^A = \underbrace{\left(1 - \frac{1}{2!} + \frac{1}{4!} + \dots\right)}_{\text{looks like cos}} I + \underbrace{\left(1 - \frac{1}{3!} + \frac{1}{5!} + \dots\right)}_{\text{looks like sin}} A$$

$$e^{KA} = (\cos K) I + (\sin K) A = \begin{pmatrix} \cos K + \sin K & \sqrt{2} \sin K \\ -\sqrt{2} \sin K & \cos K - \sin K \end{pmatrix}$$

In general, the exponential of a matrix is

$$\exp(X) = \sum_{K=0}^{\infty} \frac{X^K}{K!}$$



$$(a+c)(b+d) - ab - 2cb - dc$$

$$\cancel{ab} + ad + cb + \cancel{cd} - \cancel{cb} - 2cb - \cancel{dc}$$

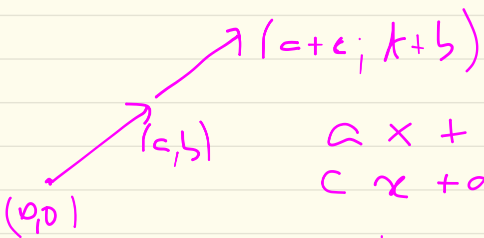
$$ad + cb - 2cb = \boxed{ad - cb}$$

$$\left.\begin{aligned} ax + by &= f \\ cx + dy &= g \end{aligned}\right\} \quad y = \left(\frac{g - cx}{d}\right) \qquad ax + b\left(\frac{g - cx}{d}\right) = f$$

$$x(ad - cb) = fd - bg$$

$$\boxed{x = \frac{1}{(ad - cb)} \cdot (fd - bg)}$$

if area is zero



$$\begin{aligned} ax + by &= f \\ cx + dy &= g \end{aligned}$$

contain same information.

$$\varepsilon_{ijk}\varepsilon_{mnp} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ip} \\ \delta_{jm} & \delta_{jn} & \delta_{jp} \\ \delta_{km} & \delta_{kn} & \delta_{kp} \end{vmatrix}$$

Null matrix

$$M = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \qquad M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_{ij}^2 = M_{ij} M_{jk}$$

$$M_{11} = M_{11} M_{11} + M_{12} M_{21} = 0 \qquad M^2 = 0 \text{ but } M \neq 0$$

5. Singular matrix $\Rightarrow$ Not invertible

$$Mx = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 4y \\ x - 2y \end{pmatrix}$$

$$M(Mx) = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2x - 4y \\ x - 2y \end{pmatrix} = \begin{pmatrix} \cancel{4x - 8y - 4x + 8y} \\ \cancel{2x - 4y - 2x + 4y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

M is an operator. If operated twice, leads to a zero amplitude vector.

$$\left.\begin{matrix} x = 1 \\ y = 0 \end{matrix}\right\} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Big\} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x = 0 \\ y = 1 \end{matrix} \quad \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Pauli matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Linear Operators

Given A and B, $aA + bB$ is a linear combination of A and B.

A function of a vector is linear if

$$f(r_1 + r_2) = f(\vec{r}_1) + f(\vec{r}_2) \quad ; \quad f(ar) = af(r)$$

In general, an operator O is linear if

$$O(A+B) = O(A) + O(B) \quad ; \quad O(KA) = K\,O(A)$$

Matrices obey this property

$$M(r_1 + r_2) = Mr_1 + Mr_2$$

$$M(Kr) = K\,M(r)$$

So, matrices represent linear operators.

Orthogonal matrices: preserve length

2×2 orthogonal matrices with determinant $=1$ corresponds to a rotation.

Determinants of orthogonal matrices are equal to ± 1

$$\langle Ae_i, Ae_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & i\neq j \end{cases}$$

$$(Av)\cdot(Av) = v\cdot v$$

If A and B are orthogonal matrices then AB is also an orthogonal matrix

$$AA^T = I \ ; \ BB^T = I$$

$$(AB)(AB)^T = AB\,B^T A^T = AIA^T = AA^T = I$$

So AB is orthogonal

$$\det(A^T) = \det(A)$$

$$1 = \det(I) = \det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A)$$
$$= (\det A)^2$$

$$\therefore \det A = \pm 1$$

In 2D, +1 means rotation, −1 means reflection

(Give examples)

$A_x = A \cdot \hat{x} \quad A_y = A \cdot \hat{y} \quad A_z = A \cdot \hat{z}$

$$\begin{pmatrix} A_{x_1} & \cdot & \cdot \\ A_{x_2} & \cdot & \cdot \\ A_{x_3} & \cdot & \cdot \end{pmatrix} \begin{pmatrix} A_{x_1} & A_{x_2} & A_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \mathbb{1}$$

Proof that $AA^T = I$ for orthonormal; $A^{-1} = A^T$

$$A = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

How to find that A is a rotation?

Identify with $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

or check how it operates on the unit vector $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

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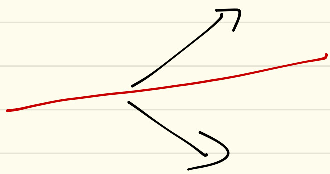
Solve for reflection



what is the line of reflection?

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ operating on a vector simply keeps x and flips y. So, the reflection plane is $y=0$.

How about $\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$?

For that we realize that through the line of reflection, a vector is mapped to itself. So, we want to find

$$Ar = r$$

What is the vector (x, y) that the matrix A maps to itself.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(a_{11}-1)x + a_{12}y = 0$$
$$a_{21}x + (a_{22}-1)y = 0$$

yields $\boxed{y = -x\sqrt{3}}$

That's the axis.

Find the mapping produced by the matrix

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$Gr = r \Rightarrow$ vector $(1, 0, 1)$ is unchanged.

So, it is a rotation about the $\hat{i} + \hat{k}$ axis

Also, G^2 is the identity matrix, so the angle of rotation is $180°$.

For K, the unchanged is $(1, -1, 1)$

$K^3 = 360$

Rotation of $-120°$

Mapping of $L = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Not a rotation. Reflection.

Solve for $Lr = -r$