Math Methods in Physics I

Prof Wladimir Lyra

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Class #18

Powers of matrices

Following eigendecomposition

$$M = C D C^{-1}$$

$$M^2 = (C D C^{-1})(C D C^{-1}) = C D C^{-1} C D C^{-1} = C D^2 C^{-1}$$

$$\boxed{M^n = C D^n C^{-1}}$$

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = \sum_{k=0}^{\infty} \frac{C D^k C^{-1}}{k!}$$

$$D_{ij} = \lambda \delta_{ij} \qquad D^2 = D_{ik} D_{kj} = \lambda^2 \delta_{ik} \delta_{kj} = \lambda^2 \delta_{ii} \delta_{ij} = \lambda^2 \delta_{ij}$$

$$D^n = \lambda^n \delta_{ij}$$

It is a simple way to find powers. Again, go back to geometric interpretation: rotate to align with ellipse axes, stretch, rotate to original orientation. The power applies the stretching several times, the rotation is unchanged.

Fibonacci numbers

$F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

0 1 1 2 3 5 8 13 21 ...

Show pop culture example, The da Vinci Code

Kepler noticed that the ratio of two Fibonacci numbers approaches the Golden ratio

Kepler's conjecture:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

where φ is the golden ratio.

Golden ratio

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi} = \varphi$$

$$\therefore \varphi + 1 = \varphi^2 \to \varphi^2 - \varphi - 1 = 0 \qquad \varphi = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1+\sqrt{5}}{2} = 1.61803\ldots$$

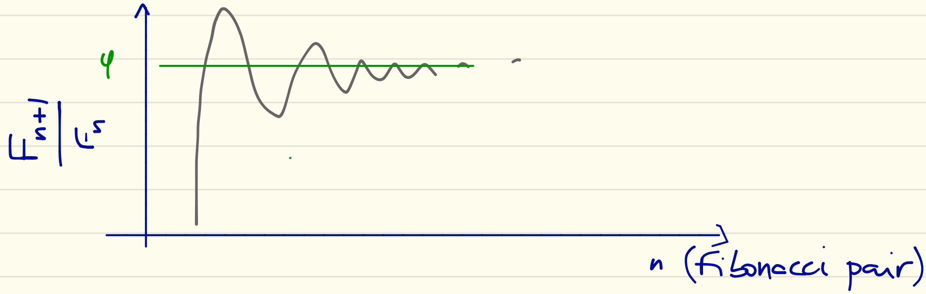
$$\frac{1-\sqrt{5}}{2} = -0.618\ldots$$

Because it is the ratio of two numbers, it must necessarily be positive

Give examples of the golden ratio in nature, architecture and art.
Beethoven spaced the 5th symphony motto in golden sections
Literary device used in classic novels: climax in golden section

between beginning and end.

Show script in python that shows that the Fibonacci series indeed goes to the golden ratio



Show geometrical interpretation as Fibonacci rectangle and spiral approach the golden rectangle and spiral.

Proof of Kepler conjecture through Linear Algebra

Write the fibonacci pairs:

$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}2\\1\end{pmatrix} \to \begin{pmatrix}3\\2\end{pmatrix} \to \begin{pmatrix}5\\3\end{pmatrix} \to \begin{pmatrix}8\\5\end{pmatrix}$$

The pattern is

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x+y\\x\end{pmatrix}$$

So, we are looking for an operator A that does

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \end{pmatrix}$$

Find A by applying it on unit vectors

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad A = \begin{pmatrix} 1 & a_{12} \\ 1 & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b=0 \quad a=1 \qquad \boxed{A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$A u_k = A\begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} = \begin{pmatrix} F_k + F_{k-1} \\ F_k \end{pmatrix} = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$$

So,

$$\boxed{\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$A = CDC^{-1} \Rightarrow A^n = CD^nC^{-1}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -(1-\lambda)\lambda - 1 = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

Eigenvalues:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Eigenvectors:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{matrix} x + y = \lambda x \\ x = \lambda y \end{matrix} \quad \left\} \begin{matrix} y = (\lambda - 1) x \\ y = \dfrac{x}{\lambda} \end{matrix} \right.$$

$x = \lambda y \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

If they doubt it:

First eigenvector:

$$y = (\lambda - 1)x \qquad \begin{cases} y_1 = (\lambda_1 - 1)x_1 \\ y_1 = \dfrac{x_1}{\lambda_1} \end{cases}$$
$$y = \frac{x}{\lambda}$$

$$\frac{1+\sqrt{5}}{2} \Rightarrow \frac{1+\sqrt{5}}{2} - \frac{2}{2} = \frac{\sqrt{5}-1}{2}$$

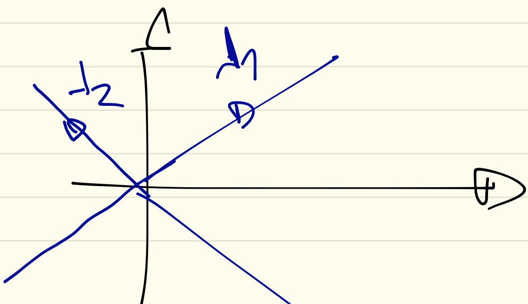
$$\frac{x_1}{\lambda_1} = \frac{\lambda_1}{(1+\sqrt{5})} \cdot 2 = \frac{2}{(1+\sqrt{5})} \cdot \frac{(\sqrt{5}-1)}{(\sqrt{5}-1)} = \frac{2(\sqrt{5}-1)}{4} = \frac{\sqrt{5}-1}{2}$$

$$y = \left(\frac{\sqrt{5}-1}{2}\right)x$$

Second eigenvector

$$\frac{1-\sqrt{5}}{2} - 1 = -\frac{\sqrt{5}-1}{2} = -\frac{(1+\sqrt{5})}{2}$$

$$y = -\frac{(1+\sqrt{5})}{2}x$$



$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad C = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \qquad C^{-1} = \frac{1}{(\lambda_1 - \lambda_2)} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Check $C^{-1}AC = \frac{1}{(\lambda_1-\lambda_2)} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} = \frac{1}{(\lambda_1-\lambda_2)} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 + 1 & \lambda_2 + 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$

$$A^n = C D C^{-1} = C \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} C^{-1}$$

So $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = C \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} C^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \frac{1}{(\lambda_1-\lambda_2)} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(\lambda_1-\lambda_2)} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix}$$ single column

$$\boxed{\begin{aligned} F_n &= \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}} \\ F_{n+1} &= \frac{\lambda_1^{(n+1)} - \lambda_2^{(n+1)}}{(\lambda_1 - \lambda_2)} \end{aligned}}$$

$$F_{n+1} = \frac{\lambda_1^{(n+1)} - \lambda_2^{(n+1)}}{(\lambda_1 - \lambda_2)} \qquad \frac{F_{n+1}}{F_n} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n}$$

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

$$\lim_{n\to\infty} \frac{\left[\frac{(1+\sqrt{5})}{2}\right]^{n+1} - \left[\frac{(1-\sqrt{5})}{2}\right]^{n+1}}{\left[\frac{1+\sqrt{5}}{2}\right]^{n} - \left[\frac{1-\sqrt{5}}{2}\right]^{n}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1+\sqrt{5})^n}$$

$$a = (1+\sqrt{5}) \;;\; b = (1-\sqrt{5})$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{a^{n+1} - b^{n+1}}{a^n - b^n} = \frac{1}{2} \lim_{n\to\infty} \frac{a a^n - b b^n}{a^n - b^n} = \frac{1}{2} \lim_{n\to\infty} \frac{(a a^n - b b^n)/a^n}{(a^n - b^n)/a^n}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{a - b b^n/a^n}{1 - b^n/a^n} = \frac{1}{2} \lim_{n\to\infty} \frac{a - b\left(\frac{b^n}{a^n}\right)}{1 - \frac{b^n}{a^n}}$$

$$\lim_{n\to\infty} \frac{b^n}{a^n} = \lim_{n\to\infty} \frac{(1-\sqrt{5})^n}{(1+\sqrt{5})^n} = 0$$

So,

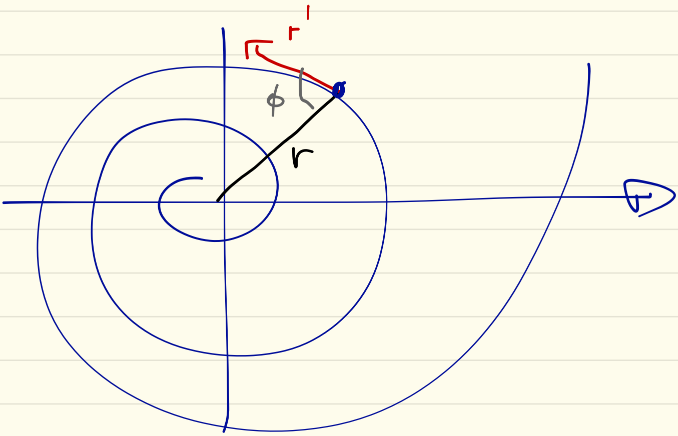
$$\boxed{\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{a}{2} = \frac{1+\sqrt{5}}{2} = 1.618\ldots}$$

Q.E.D.

Logarithmic spirals

The golden spiral is a specific case of logarithmic spirals, curves defined by

$$r = a e^{b\theta} \quad , \text{ with } a \text{ and } b \text{ constant}$$



They have the property that the pitch angle, the angle between the radial vector r and the tangent to the spiral, $r' = \frac{dr}{d\theta}$, is constant.

$$\frac{dr}{d\theta} = a b e^{b\theta} = b r$$

$$\phi : \text{pitch angle} = \tan^{-1}\left(\frac{r}{\frac{dr}{d\theta}}\right) = \tan^{-1}\left(\frac{1}{b}\right) \quad : \text{const}$$