Math Methods in Physics I

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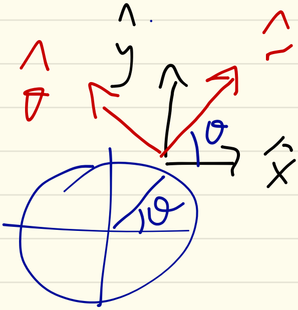
Class #9

Linear Algebra

Think about matrix multiplication. why is it $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

instead of $A_{ij} B_{ij} = C_{ij}$ where $C_{ij} = a_{ij} b_{ij}$ why not this?

Rotation

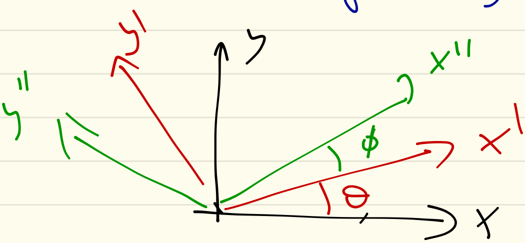


$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \cos\theta\, y$$

Define it as a Rotation Operator $R(\theta)$, so that

$$\vec{x}\,' = R(\theta)\,\vec{x}$$

Now, I want to rotate again by another angle, ϕ



The properties must be so that

$$x'' = \cos\phi\, x' + \sin\phi\, y'$$
$$y'' = -\sin\phi\, x' + \cos\phi\, y'$$

and so that $\vec{r}'' = R(\phi)\, \vec{r}'$

And that the two rotations, by θ and ϕ, are equal to a single one, of $\theta+\phi$:

$$x'' = \cos(\theta+\phi)\, x + \sin(\theta+\phi)\, y$$
$$y'' = -\sin(\theta+\phi)\, x + \cos(\theta+\phi)\, y$$

This one is obvious (show anyway). What is less obvious is that this must be equal to:

$$\vec{x}'' = R(\theta+\phi)\, \vec{x} = R(\phi)\, R(\theta)\, \vec{x}$$

What algebra satisfies these conditions?

Let's pass to general statements

$$x' = ax + by \qquad \text{Now} \qquad x'' = a'x' + b'y'$$
$$y' = cx + dy \qquad \qquad \qquad y'' = c'x' + d'y'$$

Express x'' and y'' in terms of x and y

$$x'' = a'(ax+by) + b'(cx+dy)$$
$$y'' = c'(ax+by) + d'(cx+dy)$$

$$x'' = (a'a + b'c)x + (a'b + b'd)y$$
$$y'' = (c'a + d'c)x + (c'b + d'd)y$$

$$\boxed{\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}} \qquad \boxed{\begin{aligned} x'' &= a'x' + b'y' \\ y'' &= c'x' + d'y' \end{aligned}}$$

$$\boxed{\begin{aligned} x'' &= (a'a + b'c)x + (a'b + b'd)y \\ y'' &= (c'a + d'c)x + (c'b + d'd)y \end{aligned}}$$

$$r' = Ar$$
$$r'' = Br' \qquad \therefore \; r'' = B(Ar) = Cr$$

$$x'' = (a'a + b'c)x + (a'b + b'd)y = \alpha x + \beta y$$
$$y'' = (c'a + d'c)x + (c'b + d'd)y = \gamma x + \delta y$$

$$\alpha = a'a + b'c \qquad \beta = a'b + b'd$$
$$\gamma = c'a + d'c \qquad \delta = c'b + d'd$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$x = r_1 \qquad x' = r_1' \qquad r_1' = a r_1 + b r_2 = a_{11} r_1 + a_{12} r_2 = \sum_j a_{1j} r_j$$
$$y = r_2 \qquad y' = r_2' \qquad r_2' = c r_1 + d r_2 = a_{21} r_1 + a_{22} r_2 = \sum_j a_{2j} r_j$$

$$\boxed{r_i' = \sum_j a_{ij} r_j}$$

$$r'' = B r' \rightarrow r_i'' = \sum_j b_{ij} r_j'$$
$$r_i'' = \sum_j b_{ij} \left(\sum_k a_{jk} r_k \right) = \sum_j \sum_k b_{ij} a_{jk} r_k$$

$$r_i'' = \sum_j b_{ij} \left(\sum_k a_{jk} r_k \right) = b_{ij} a_{jk} r_k = c_{ip} r_p$$

$$\boxed{c_{ik} = b_{ij} a_{jk}} \qquad \boxed{C = BA = \sum_j b_{ij} a_{jk}}$$

Matrix Multiplication of transformation. ST must be the product of S times T.

Matrix multiplication is just a bookkeeping device for systems of linear substitutions plugged into one another. The formulas are not intuitive, but it's nothing other than the simple idea of combining two linear changes of variables in sucession.

Matrix multiplication in index notation

$$(AB)_{ij} = A_{ik} B_{kj}$$

$$(BA)_{ij} = B_{ik} A_{kj}$$

different matrices. - Operation doesn't commute

They always warn you that you have to be careful when multiplying matrices because $AB \neq BA$. This is thrown as some black magic not done quite right. You know there's this algebra we do with these squares of numbers, but it doesn't quite add up, so just remember that $AB \neq BA$ so you don't screw up on the exam. What they don't say is that MATRIX MULTIPLICATION IS A TRANSFORMATION. PRODUCT OF TWO MATRICES ARE TWO TRANSFORMATIONS IN SUCESSION. THE ORDER MATTERS!!

If I have

Example A = turn left B = Walk three steps

Back to rotation

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ Rotation matrix in 2D

$\vec{r} = R \cdot x$ $\qquad r_i = R_{ij} x_j$

Verify that two rotations, by θ and then by ϕ, will lead to a single rotation $\theta + \phi$

$$\vec{r} = R(\phi)\left[R(\theta)\hat{x}\right]$$

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\theta & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{bmatrix}$$

$$a_{11} = (\cos\phi\cos\theta - \sin\phi\sin\theta)\hat{x} + (\cos\phi\sin\theta + \sin\phi\cos\theta)\hat{y}$$

$$= \cos(\theta+\phi)\hat{x} + \sin(\theta+\phi)\hat{y}$$

$$a_{21} = -\sin\theta\cos\theta - \sin\theta\cos\phi\,\hat{x} \quad -\sin\theta\sin\theta + \cos\phi\cos\theta\,\hat{y}$$

$$= -\sin(\theta+\phi)\hat{x} + \cos(\theta+\phi)\hat{y}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

$$R(\phi)R(\theta)\vec{x} = R(\phi+\phi)\hat{x}$$

but $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} =$

$$\begin{bmatrix} \cos\theta\cos\phi - \sin\phi\sin\theta & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \quad \therefore \; R(\theta)R(\phi) = R(\theta+\phi)$$

Use Igo example now. Show how

Polar coordinates = $R(\theta)$ Cartesian coordinates

$$\vec{r} = R(\theta)\vec{x} \qquad \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$