Math Methods in Physics I

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Nov 22th, 2016

Class #25

Parseval Theorem

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$$

Square and average over $-\pi, \pi$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx$$

Square of sine or co-sine over period is 1/2

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (a_0/2)^2 dx = (a_0/2)^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (a_n \cos nx)^2 dx = \frac{a_n^2}{2\pi}\int_{-\pi}^{\pi} (\cos nx)^2 dx = \frac{a_n^2}{2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (b_n \sin nx)^2 dx = \frac{b_n^2}{2\pi}\int_{-\pi}^{\pi} (\sin nx) dx = \frac{b_n^2}{2}$$

The cross terms with $\sin nx \sin mx$ or $\cos nx \cos mx$ or $\sin nx \cos mx$ all zero

So:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2$$

Unchanged if period is different

$$\frac{1}{2L}\int_{-L}^{L} [f(x)]^2 dx = \sum_{-\infty}^{\infty} |c_n|^2$$

Fourier Transform

Discrete

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx\pi/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$$

Discrete set of frequencies. $\frac{n}{2L}$

for a continuous function

$$f(x) = \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk$$

Continuous set of frequencies

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

Non-periodic function. Let the period go from $[-L, L]$ to $[-\infty, \infty]$.

$k = \frac{n\pi}{L}$ and $k_{n+1} - k_n = \frac{\pi}{L} = \Delta k$, then

$$\frac{1}{2L} = \frac{\Delta \alpha}{2\pi}$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{ik_n x}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-ik_n x}\, dx = \frac{\Delta k}{2\pi}\int_{-L}^{L} f(u)\, e^{-ik_n u}\, du$$

$$f(x) = \sum_{-\infty}^{\infty}\left[\frac{\Delta k}{2\pi}\int_{-L}^{L} f(u)\, e^{-ik_n u}\, du\right] e^{ik_n x}$$

$$= \sum_{-\infty}^{\infty} \frac{\Delta k}{2\pi} \int_{-L}^{L} f(u) e^{ik_n(x-u)} du = \frac{1}{2\pi} \sum_{-\infty}^{\infty} F(k_n) \Delta k$$

where

$$F(k_n) = \int_{-L}^{L} f(u) e^{ik_n(x-u)} du$$

$\sum_{-\infty}^{\infty} F(k_n) \Delta k$ looks like the formula in calculus for which as the sum as the limit $\Delta k \Rightarrow 0$ is an integral. This is equivalent to $\Delta k = \pi/L \Rightarrow 0$, ie, the period going to infinity.

$$\lim_{\Delta k \to 0} \sum_{-\infty}^{\infty} F(k_n) \Delta k = \int_{-\infty}^{\infty} F(k) dk$$

$$F(k) = \int_{-\infty}^{\infty} f(u) e^{ik(x-u)} du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{ik(x-u)} du \, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} f(u) e^{-iku} du$$

Define $g(k)$

$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-iku} du$$

$$f(x) = \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk$$

Different notations. The product of constants of $f(x)$ and $g(k)$ must be $1/2\pi$ will be either $f(x)$, $g(k)$, or symmetrically with $1/\sqrt{2\pi}$ on both.

Parseval theorem for Fourier transforms

$$\bar{g}_1(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}_1(x)\, e^{ikx}\, dx$$

Multiply by $g_2(k)$ and integrate in k

$$\int_{-\infty}^{\infty} \bar{g}_1(k)\, g_2(k)\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \bar{f}_1(x)\, e^{ikx}\, dx \right] g_2(k)\, dk$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}_1(x)\, dx \left[\int_{-\infty}^{\infty} g_2(k)\, e^{ikx}\, dk \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}_1(x)\, f_2(x)\, dx$$

$$\int_{-\infty}^{\infty} \bar{g}_1(k)\, g_2(k)\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}_1(x)\, f_2(x)\, dx$$

Set $f_1 = f_2 = f$ and $g_1 = g_2 = g$

$$\int_{-\infty}^{\infty} |g(k)|^2\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Parseval theorem

Think of a vector V. As seen in coordinate system S with basis vector $\hat{e}_i$, it can be written

$$V = \sum_i V_i \hat{e}_i$$

where V_i are the components of V in S. As seen from another coordinate system S' with basis vectors $\hat{e}'$, it has a representation

$$V = \sum_i V_i' \hat{e}_i'$$

Obviously the length of the vector is independent of the coordinate system used to represent it. In other words, we must have

$$\sum_i V_i^2 = \sum_i (V_i')^2$$

Proceeding with the analogy, for a function $f(x)$ one can have a position space representation in δ-function basis as

$$f(x) = \int f(x') \, \delta(x - x') \, dx$$

where the "component" of $f(x)$ along the "basis vector" $\delta(x-x')$ is $f(x')$ and we sum over all the possible axes. One can look at the same function in Fourier space representation as

$$f(x) = \int g(k) \, e^{-ikx} \, dk$$

where e^{-ikx} are the "basis vectors" and $g(k)$ are the components of $f(x)$. along these basis vectors. You would then agree that

$$\int |f(x)|^2 dx = \int |g(k)|^2 dk$$

So, Parceval theorem is just the restatement of the invariance of the length of a vector, independent of the representation used.

In our case it means that the energy in real space is equal to the energy in Fourier space.

Transform unitary : $f(x) = 1$, $g(k) = \delta(k)$

Transform of sin or cos $f(x) = e^{iax}$; $g(k) = \delta(k-a)$

Transform of the box function : sinc function

Harmonic Oscillator

$$\frac{d^2y}{dt^2} + ay = 0$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega$$

$$f'(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\, \frac{d}{dt}\left(e^{i\omega t}\right) d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\, i\omega\, e^{i\omega t}\, d\omega$$

Now invert

$$F(f(t)) = g(\omega)$$

$$F(f'(t)) = i\omega g(\omega)$$

In general $F\left(\frac{d^n f(x)}{dx^n}\right) = (i\omega)^n F(f(x))$

Notation $F(f(x)) = \hat{f}(\omega)$

$$\frac{d^2y}{dt} + ay = 0$$

$$-\omega^2 g(\omega) + a g(\omega) = 0$$

$$g(\omega)\left[a - \omega^2\right] = 0$$

$$\omega^2 = a$$

$$\omega = \sqrt{a}$$

$f(x) = e^{i\omega t}$; $e^{\sqrt{a}t}$ $f(t) = A\cos\sqrt{a}t + Bi\sin\sqrt{a}t$

Sound waves

$$\frac{\partial \rho}{\partial t} + (u \cdot \nabla)\rho = -\rho \nabla \cdot u \qquad p = \rho c_s^2$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho}\nabla p$$

$$\frac{\partial \rho}{\partial t} + (u \cdot \nabla)\rho = -\rho \nabla \cdot u \qquad p = \rho \cdot \alpha \qquad (\alpha = c_s^2)$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p$$

$$\frac{\partial \rho}{\partial t} + u \cdot \frac{\partial}{\partial x} \rho = -\rho \frac{\partial}{\partial x} u$$

$$\frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\rho = \bar{\rho} + \rho' \qquad u = \bar{u} + u' \qquad p = \bar{p} + p'$$

$$e^{i(\omega t + Kx)}$$

$$\frac{\partial^2 \rho'}{\partial t^2} = -\bar{\rho} \frac{\partial}{\partial x} \frac{\partial u'}{\partial t} = -\bar{\rho} \frac{\partial}{\partial x}\left(-\frac{1}{\bar{\rho}} c_s^2 \cdot \frac{\partial \rho}{\partial x}\right)$$

$$\frac{\partial \rho'}{\partial t} = -\bar{\rho} \frac{\partial}{\partial x} u' \qquad \frac{\partial^2 \rho'}{\partial t^2} = \alpha \frac{\partial \rho'}{\partial x^2}$$

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho} \frac{\partial p'}{\partial x} \qquad p' = \rho' c_s^2 \qquad c_s^2 = \alpha \quad \text{Sound speed}$$

$$i\omega \hat{\rho} = -\bar{\rho}\, i k \hat{u} \qquad \to \qquad \hat{u} = -\frac{\omega}{\bar{\rho} K} \hat{\rho}$$

$$i \omega \hat{u} = -\frac{c_s^2}{\rho} i K \hat{\rho}$$

$$\therefore \; -\frac{\omega^2}{\cancel{\rho} K} \cancel{\hat{\rho}} = -\frac{c_s^2 k}{\rho} \cancel{\hat{\rho}} \qquad \omega^2 = c_s^2 K^2 \qquad \boxed{\omega = c_s \cdot K}$$

$$\frac{\omega^2}{K^2} = c_s^2$$