Math Methods in Physics I

Prof Wladimir Lyra

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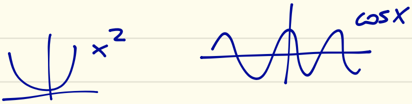
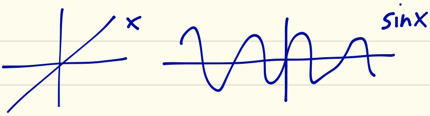
Class #24

Even and odd functions

Even $f(-x) = f(x)$



Odd $f(-x) = -f(x)$



Even × Even = Even
Odd × odd = Even
odd × Even = odd

Any function can be written in terms of a sum of an even and an odd functio

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{\text{even}} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{\text{odd}}$$

swap x for $-x$

Example:

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right) = \cosh x + \sinh x$$

works also for complex functions

$$e^{ix} = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) + \frac{1}{2}\left(e^{ix} - e^{-ix}\right)$$

$$e^{ix} = \cos x + \frac{i}{i}\left[\frac{1}{2}\left(e^{ix} - e^{ix}\right)\right] = \cos x + i\sin x$$

Integrals of even/odd functions over the interval

$$\int_{-L}^{L} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^L f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$$

Coefficients simplify. If $f(x)$ is odd. Sines are odd and cosines even. So,

$f(x)\sin\left(\frac{n\pi x}{L}\right)$ is even

$f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd

So, a_n is integral of odd function. a_n are all zero. b_n is even, twice the integral over half-period

$$f(x) \text{ for odd } \begin{cases} b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \\ a_n = 0 \end{cases}$$

$f(x)$ odd is expanded in sine series

If $f(x)$ is even, $f(x)\sin$ is odd, $f(x)\cos x$ is even, so the sines will cancel in integration, and

$$f(x) \text{ even } \begin{cases} a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = 0 \end{cases}$$

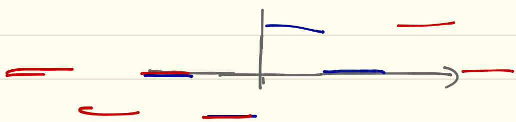
$f(x)$ even is expanded on cosine series

Do the example from book

$$f(x) = \begin{cases} 1 & ; \ 0 < x < 1/2 \\ 0 & ; \ 1/2 < x < 1 \end{cases}$$

In sine series/ cosine series / exponential

Sine series. Make it odd



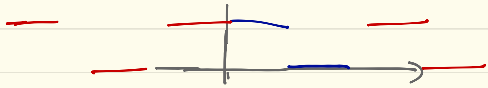
The period is 2. Odd function. $a_n = 0$

$$b_n = 2\int_0^1 f(x)\sin n x\pi\, dx = 2\int_0^{1/2} \sin n\pi x\, dx = \frac{-2}{n\pi}\cos n\pi x \Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$$

$b_1 = \frac{2}{\pi}$; $b_2 = \frac{4}{2\pi}$; $b_3 = \frac{2}{3\pi}$; $b_4 = 0$, ...

$$f(x) = \frac{2}{\pi}\left(\sin \pi x + \sin 2\pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{\sin 6\pi x}{3} + \dots\right)$$

Sketch in cosine series. Make it even



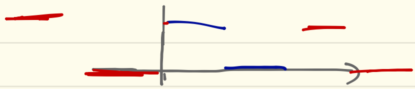
All $b_n = 0$; period 2

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = \frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\pi x - \frac{\cos 3\pi x}{3} + \frac{\cos 5\pi x}{5} + \dots\right)$$

Do the general exponential. Just repeat



Period 1

$$c_n = \int_0^1 f(x)\,e^{-2in\pi x}\,dx = \int_0^{1/2} e^{-2in\pi x}\,dx$$

$$= \frac{1 - e^{-in\pi}}{2in\pi} = \frac{1-(-1)^n}{2in\pi} = \begin{cases} \frac{1}{in\pi} ; & n \text{ odd} \\ 0 ; & n \text{ even} \neq 0 \end{cases}$$

$$C_0 = \int_0^{1/2} dx = \frac{1}{2}$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[e^{2i\pi x} - e^{-2i\pi x} + \frac{1}{3}e^{6i\pi x} - \frac{1}{3}e^{-6\pi i x} + \dots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin 2\pi x + \frac{\sin 6\pi x}{3} + \dots\right]$$

Gibbs phenomena

Gibbs phenomenon - The wiggling won't go away. It won't satisfy the Dirichlet conditions, will it? It doesn't converge to either the value or the half of it.

Give examples in signal processing.

Box function of jump $\pi/2$ at $x=0$, from $-\pi/4$ to $\pi/4$

for $x=0$

$$S_N f(0) = 0 = \left(-\frac{\pi}{4} + \frac{\pi}{4}\right)/2 = \frac{f(0^-) + f(0^+)}{2}$$

$$S_N f(x) = \sin x + \frac{1}{3}\sin 3x + \dots + \frac{1}{N-1}\sin((N-1)x)$$

$$= \sum_{K=1}^{N-1} \frac{\sin Kx}{K}$$

$$\int_a^b f(x)\,dx \approx h \sum_{k=0}^{N-1} f(x_k)$$

$$\boxed{a, b, h \,? \qquad h = \frac{(b-a)}{N}}$$

$$S_N f\left(\frac{2\pi}{2N}\right) = \sin\left(\frac{\pi}{N}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{N}\right) + \dots \frac{1}{N-1}\sin\left(\frac{(N-1)\pi}{N}\right)$$

$$= \sum_{k=1}^{N-1} \frac{\sin k\pi/N}{k} \rightarrow \text{Rectangle integration}$$

Introduce the function $\operatorname{sinc} x = \dfrac{\sin \pi x}{\pi x}$ $\qquad \dfrac{\sin(k\pi/N)}{k\cdot\pi/N} = \operatorname{sinc}\left(\dfrac{k}{N}\right)$

So,

$$\sum_{k=1}^{N-1} \frac{\sin(k\pi/N)}{k} = \frac{\pi}{N}\sum_{k=1}^{N-1} \frac{\sin(k\pi/N)}{k\cdot\pi/N} = \frac{\pi}{N}\sum_{k=1}^{N-1}\operatorname{sinc}\left(\frac{k}{N}\right)$$

$$= \frac{\pi}{2}\cdot\frac{2}{N}\sum_{k=1}^{N-1}\operatorname{sinc}\left(\frac{k}{N}\right)$$

$$\frac{2}{N}\sum_{k=1}^{N-1}\operatorname{sinc}\left(\frac{k}{N}\right) = \frac{(b-a)}{N}\int_a^b \operatorname{sinc}\left(\frac{x}{N}\right)dx$$

$t = \dfrac{x}{N} \qquad dt = \dfrac{dx}{N}$

$$= \frac{(b-a)}{\cancel{N}}\int_a^b \operatorname{sinc}(t)\,dt\cdot\cancel{N} = \int_0^1 \operatorname{sinc} x\,dx$$

So, $$\lim_{N\to\infty} S_N f\left(\frac{\pi}{N}\right) = \frac{\pi}{2}\int_0^1 \operatorname{sinc} x\,dx$$

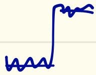
$$= \frac{1}{2}\int_0^\pi \frac{\sin x}{x}dx = \frac{\pi}{4} + \frac{\pi}{2}\cdot 0.0894898\dots$$

On the left side

$$\lim_{N\to-\infty} S_N f\left(-\frac{\pi}{N}\right) = -\frac{\pi}{4} - \frac{\pi}{2}\cdot 0.0894898$$

left limit $f(x_0^-)$ right limit $f(x_0^+)$ non-zero gap a

$f(x_0^+) - f(x_0^-) = a$



$S_N f(x)$ partial fourier series

$$= \sum_{-N<x<N} \hat{f}(n) e^{\frac{2i\pi n x}{L}} = \frac{a_0}{2} \sum_{n=1}^{N} \left(a_n \cos \frac{2\pi n x}{L} + b_n \sin \frac{2\pi n x}{L} \right)$$

$$\hat{f}(n) = \frac{1}{L} \int_0^L f(x) e^{-2i\pi n x/L} \, dx$$

$$a_n = 2/L \int_0^L f(t) \cos \frac{2\pi n x}{L} \, dx$$

$$b_n = \frac{2}{L} \int_0^l f(t) \sin \frac{2\pi n x}{L} \, dx$$

$$\lim_{N\to\infty} S_N f\left(x_0 + \frac{L}{2N}\right) = f(x_0^+) + a \cdot (0.08948987\ldots)$$

$$\lim_{N\to\infty} S_N f\left(x_0 - \frac{L}{2N}\right) = f(x_0^-) - a(0.0894\ldots)$$

but $$\lim_{N\to\infty} S_N f(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2}$$