Math Methods in Physics I

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Class # 6

Logarithms of complex numbers

$$w = \ln z = \ln re^{i\theta} = \ln r + i\theta$$

$$\ln i = \ln e^{i\pi/2} = \frac{i\pi}{2}$$

$$\ln -1 = \ln e^{i\pi} = i\pi$$

$$e^{i\pi} + 1 = 0 \rightarrow e^{i\pi} = -1 \longrightarrow \ln(-1) = i\pi$$

Complex powers

$$\boxed{w = i^i}$$

$$\ln w = i \cdot \ln i = i \ln\left(e^{i\pi/2}\right) = -\frac{\pi}{2}$$

$$\boxed{i^i = w = e^{-\pi/2}}$$

i^i is a real number

hint after that
there are
several solutions

$i = e^{i\pi/2}$

$\therefore i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2} \approx 0.2 \approx \frac{1}{5}$

$i^i \approx \frac{1}{5}$

But also $i = e^{i(\pi/2 \pm 2n\pi)}$

so, $i^i = e^{-\frac{\pi}{2} - 2n\pi} = e^{-\frac{(1\pm 4n)\pi}{2}} = e^{-\frac{\pi}{2}}, e^{-\frac{5\pi}{2}}, e^{-\frac{9\pi}{2}}, e^{-\frac{13\pi}{2}}$

$e^{\frac{3\pi}{2}}, e^{\frac{7\pi}{2}}, e^{\frac{11\pi}{2}}$

$i^{2i} = \left(i^i\right)^2 = \left(e^{-\pi/2}\right)^2 = e^{-\pi}$

$= \left[e^{-\frac{(1\pm 4n)\pi}{2}}\right]^2 = e^{-(1\pm 4n)\pi} = e^{-\pi}, e^{-5\pi}, e^{3\pi}$

$i^{-2i} = \left(i^i\right)^{-2} = \left(e^{-\pi/2}\right)^{-2} = e^{\pi} \quad \left(e^{5\pi}, e^{-3\pi}, \ldots\right)$

What is this? does it mean that $e^{-\pi} = e^{3\pi}$?

Nope. This is like when we write $\sqrt{4} = \pm 2$, and obviously

$-2 \neq 2$. But this illustrates the situation with imaginary numbers is not unlike that with negative numbers.

Why is it that $(-1)(-1) = 1$? One can say that to multiply by -1 is equal to "revert" the magnitude in the real axis. In this case, in the complex plane, to multiply by i is to rotate forward by $90°$.

Tell a bit more of the history of complex numbers.

Application..

$$m\frac{d^2x}{dt^2} = -Kx$$ harmonic oscillator

$$X = Ae^{i\omega t}$$ (ansatz)

$$x' = Ai\omega e^{i\omega t}$$

$$x'' = -A\omega^2 e^{i\omega t}$$

$$\therefore -\cancel{A}\omega^2 \cancel{e^{i\omega t}} = -\frac{K}{m}\cancel{A e^{i\omega t}}$$

$$\omega = \sqrt{\frac{K}{m}}$$

Book asks them to memorize, to problem in your head. I need to tell them that it is not needed to memorize anything or do anything in one's head.

$\cos z = 2$; $\arccos 2$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 2$$

$u = e^{iz}$; $e^{-iz} = 1/u$

$\therefore$ $u + 1/u = 4 \rightarrow u^2 - 4u + 1 = 0$

$$u = \frac{4 \mp \sqrt{16-4}}{2} = 2 \pm \sqrt{3} \qquad e^{iz} = u = 2 \pm \sqrt{3}$$

Take log on both sides

$$iz = \log u = \log(2 \pm \sqrt{3}) = \operatorname{Ln}(2 \pm \sqrt{3}) + 2n\pi i$$

$$z = 2n\pi - i \operatorname{Ln}(2 \pm \sqrt{3})$$

Some notation

$$w = \ln z = \ln\left(re^{i(\theta \pm 2\pi n)}\right) = \ln r + i\theta \pm 2\pi n$$

$\operatorname{Ln} z \equiv$ principal value of $\ln z$, for which $0 \leq \theta \leq 2\pi$

$\ln$ is an operator

Linear Algebra

Combination of algebra and geometry.

Give a brief refresher on matrices

Some useful matrices
The rotation matrix

Example 4 → Series of matrices

Words of warning $(A+B)^2 = A^2 + AB + BA + B^2$
(Don't write $2AB$)

$e^{A+B} \neq e^A \cdot e^B$ if A, and B don't commute.

Do prob 29 and Probl 15.34

Recap of matrix operations

$$\begin{aligned} 2x \quad\quad - z &= 2 \\ 6x + 5y + 3z &= 7 \\ 2x - y \quad\quad &= 4 \end{aligned} \qquad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 6 & 5 & 3 \\ 2 & -1 & 0 \end{pmatrix}$$

$$Ax = b$$

or

$$\sum_{j=1}^{3} A_{ij} x_j = b_i$$

Einstein notation

$$\sum_{j=1}^{3} A_{ij} x_j = b_i \quad \Rightarrow \quad A_{ij} x_j = b_i$$

Omit summations. Sum over repeated indices apply.

Applications. Write equations in component form

$$\nabla \psi = \partial_i \psi$$

$$\nabla \cdot A = \partial_i A_i$$

$$\nabla \times A = (\partial_j A_k - \partial_k A_j)_i$$

$$\nabla^2 A = \partial_{jj} A_i$$

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Row reduction (Gauss elimination)

$$\begin{aligned} 2x \quad\quad - z &= 2 \quad (A) \\ 6x + 5y + 3z &= 7 \quad (B) \\ 2x - y \quad\quad &= 4 \quad (C) \end{aligned}$$

$$\boxed{\begin{aligned} B' &= B - \frac{b_1}{a_1} A \quad \text{(zero the x coeffs)} \\ C' &= C - \frac{c_1}{a_1} A \end{aligned}}$$

$$d_2' = d_2 - \frac{d_1}{a_1} A$$

$$\begin{aligned} \cancel{6x} + 5y + 3z &= 7 \\ - \cancel{6x} \quad\quad + 3z &= -6 \\ \hline 5y + 6z &= 1 \end{aligned}$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\begin{aligned} 2x \quad\quad\quad - z &= 2 \quad (A) \\ 5y + 6z &= 1 \quad (B') \\ 2x - y \quad\quad &= 4 \quad (C) \end{aligned}$$

$c' = c - \frac{c_1}{a_1} A$

$d_3' = d_3 - \frac{d_1}{a_1} A$

$$\begin{array}{r} 2x - y \quad = 4 \\ -2x \quad + z = -2 \\ \hline -y + z = 2 \end{array}$$

Exchange B and C

$$\begin{array}{lr} A & 2x \quad\quad - z = 2 \\ B' & 5y + 6z = 1 \\ c' & -y + z = 2 \end{array}$$

$c'' = c' - \frac{b_2'}{c_2'} B$

$d'' = d' - \frac{b_2'}{d_2'} B$

$$\begin{array}{r} \cancel{5y} + 6z = 1 \\ -\cancel{5y} + 5z = 10 \\ \hline 11z = 11 \rightarrow z = 1 \end{array}$$

$$\begin{aligned} 2x \quad - z &= 2 \\ -y + z &= 2 \\ z &= 1 \end{aligned}$$

Eliminate z $2x \quad = 3$

$-y = 1$

$z = 1$

$$\Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1 \\ 1 \end{pmatrix}$$

Faster: "Matrix augmentation"

$$\begin{pmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{pmatrix} \times -3 \quad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & 5 & 6 & 1 \\ 2 & -1 & 0 & 4 \end{pmatrix} \times -1 \quad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & 5 & 6 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix} \text{ SWAP}$$

$$\begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 5 & 6 & 1 \end{pmatrix} \times 5 \quad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 11 & 11 \end{pmatrix} /11 \quad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \times -1$$

$$\begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \times 1 \quad \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} /2 \\ \times -1 \\ \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1 \\ 1 \end{pmatrix}$$

Kramer rule

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \qquad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

$$\begin{pmatrix} c_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} . \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$Ax = b$$

$$x_i = \frac{\det(A_i)}{\det(A)}$$

A_i is the matrix formed by replacing the i-th column of A by the column vector b.

3x3 systems

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$x = \frac{1}{D} \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad y = \frac{1}{D} \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad z = \frac{1}{D} \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

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$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$Ar = b \qquad r = \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\gamma x - \gamma v t = x'$$
$$\therefore -\frac{\gamma v}{c^2} x + \gamma t = t'$$

$$\begin{pmatrix} \gamma & -\gamma v \\ -\frac{\gamma v}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x' \\ t' \end{pmatrix} \qquad \therefore x = \frac{\begin{vmatrix} x' & -\gamma v \\ t' & \gamma \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{vmatrix}}$$

$$x = \frac{\begin{vmatrix} x' & -\gamma v \\ t' & \gamma \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{vmatrix}} \qquad t = \frac{\begin{vmatrix} \gamma & x' \\ -\gamma v/c^2 & t' \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{vmatrix}}$$

$$\begin{vmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{vmatrix} = \gamma^2 - \gamma^2 v^2/c^2 = \gamma^2\left(1 - v^2/c^2\right) = 1$$

$$\begin{vmatrix} x' & -\gamma v \\ t' & \gamma \end{vmatrix} = x'\gamma + \gamma v t' = \gamma\left(x' + v t'\right)$$

$$\begin{vmatrix} \gamma & x' \\ -\gamma v/c^2 & t' \end{vmatrix} = \gamma t' + \gamma x' v/c^2 = \gamma\left(t' + \frac{x' v}{c^2}\right)$$

$$\therefore \quad x = \gamma\left(x' + v t'\right)$$
$$t = \gamma\left(t' + \frac{x' v}{c^2}\right)$$