Math Methods in Physics I

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class #5

Try some more roots of complex numbers

$$z = \sqrt[6]{-8i}$$

$$w = -8i \quad \therefore \quad w = re^{i\theta} = 8e^{i\left(\frac{3\pi}{2} + 2k\pi\right)}$$

$$r = \sqrt[6]{w} = \sqrt[6]{8}\, e^{i\left(\frac{3\pi}{2} + 2k\pi\right)/6}$$

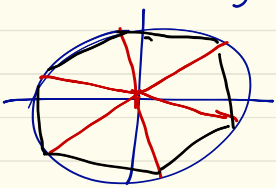
$$= 2^{3/6}\, e^{i\left(\frac{3\pi}{12} + \frac{2k\pi}{6}\right)} = 2^{1/2}\, e^{i\left(\frac{\pi}{4} + \frac{k\pi}{3}\right)} \qquad k = 0, 1, 2, 3, 4, \ldots$$

$$r = 2^{1/2} \cdot \exp\left(i\left[\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}\right]\right) \qquad \frac{\pi}{12} = \frac{\tau}{24} = \frac{360}{24} = \frac{\cancel{12} \cdot 30}{2 \cdot \cancel{12}}$$

$$= 15 \text{ degrees}$$

$$\frac{3\pi}{12} + \frac{4\pi}{12} \; i$$

Start at 45, go doing jumps of 60 degrees



Show drawings.

Trigonometric functions of complex numbers

$$\boxed{\cos i \; ?}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

So,
$$\cos i = \frac{e^{-1} + e}{2} \approx 1.543$$

Hyperbolic functions

Try with these functions $\left(\frac{e^{ix} + e^{-ix}}{2}\right)$ and $\left(\frac{e^{ix} - e^{ix}}{2i}\right)$

ix appears in both. Replace $x \to iz$

$$\cos(iz) = (e^{-z} + e^{z})/2 = (e^{z} + e^{-z})/2$$
$$\sin(iz) = (e^{-z} - e^{z})/2i = i(e^{z} - e^{-z})/2$$

$$\cos(iz) = f(z)$$
$$\sin(iz) = i g(z)$$

$$\cos^2(z) + \sin^2(z) = 1 \quad \text{CIRCLE}$$
$$f^2(z) + (i g(z))^2 = 1$$

$$f^2(z) - g^2(z) = 1 \quad \text{HYPERBOLA}$$

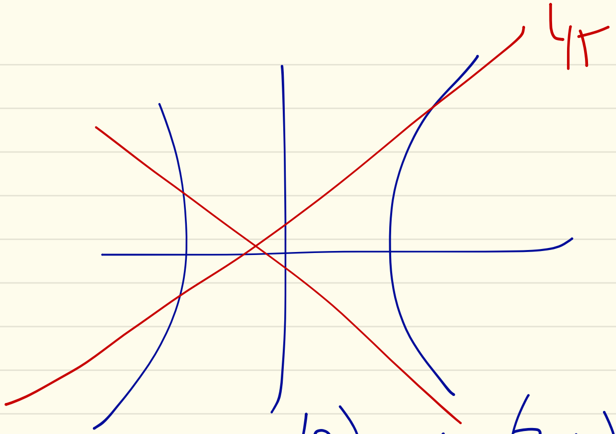
$f(z) = \cosh(z)$ hyperbolic 1- -

$g(z) = \sinh(z)$ hyperbolic sine

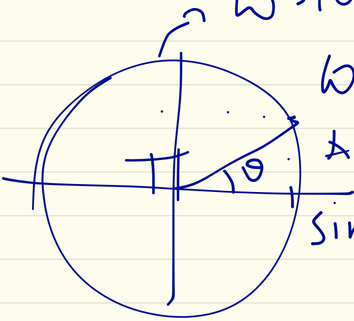
$$\sinh z = \frac{e^z - e^{-z}}{2}; \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

From the definition:

$$\sin iz = i \sinh z$$
$$\cos iz = \cosh z$$



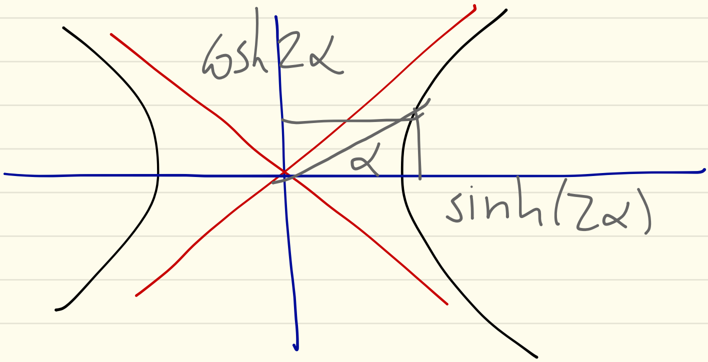
$\cos(\theta) = \cos(2\alpha)$



area $\alpha = \theta/2$

$\sin(\theta) = \sin(2\alpha)$

Now, think of area under hyperbola



Logarithms of complex numbers

$$w = \ln z = \ln re^{i\theta} = \ln r + i\theta$$

$$\therefore \ln i = \ln e^{i\pi/2} = \frac{i\pi}{2}$$

$$\ln -1 = \ln e^{i\pi} = i\pi$$

$$e^{i\pi} + 1 = 0 \rightarrow e^{i\pi} = -1 \rightarrow \ln(-1) = i\pi$$

Complex powers

$$\boxed{w = i^i}$$

$$\ln w = i \cdot \ln i = i \ln\left(e^{i\pi/2}\right) = -\frac{\pi}{2}$$

$$\boxed{i^i = w = e^{-\pi/2}}$$

i^i is a real number

hint after that
there are
several solutions

$i = e^{i\pi/2}$

$\therefore \; i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2} \approx 0.2 \approx \frac{1}{5}$

$i^i \approx \frac{1}{5}$

But also $i = e^{i(\pi/2 \pm 2n\pi)}$

so, $\boxed{i^i = e^{-\frac{\pi}{2} - 2n\pi} = e^{-\frac{(1 \pm 4n)\pi}{2}} = e^{-\frac{\pi}{2}}, e^{-\frac{5\pi}{2}}, e^{-\frac{9\pi}{2}}, e^{-\frac{13\pi}{2}}, e^{\frac{3\pi}{2}}, e^{\frac{7\pi}{2}}, e^{\frac{11\pi}{2}},}$

$i^{2i} = \left(i^i\right)^2 = \left(e^{-\pi/2}\right)^2 = e^{-\pi}$

$= \left[e^{-\frac{(1 \pm 4n)\pi}{2}}\right]^2 = e^{-(1 \pm 4n)\pi} = e^{-\pi}, e^{-5\pi}, e^{3\pi}$

$i^{-2i} = \left(i^i\right)^{-2} = \left(e^{-\pi/2}\right)^{-2} = e^{\pi} \; \left(e^{5\pi}, e^{-3\pi}, \ldots\right)$

What is this? does it mean that $e^{-\pi} = e^{3\pi}$?

Nope. This is like when we write $\sqrt{4} = \pm 2$, and obviously

$-2 \neq 2$. But this illustrates the situation with imaginary numbers is not unlike that with negative numbers.

Why is it that $(-1)(-1) = 1$? One can say that to multiply by -1 is equal to "revert" the magnitude in the real axis. In this case, in the complex plane, to multiply by i is to rotate forward by $90°$.

Application..

$$m\frac{d^2x}{dt^2} = -Kx$$ harmonic oscillator

$$x = Ae^{i\omega t}$$ (ansatz)

$$x' = Ai\omega\, e^{i\omega t}$$

$$x'' = -A\omega^2 e^{i\omega t}$$

$$\therefore\ -\cancel{A}\,\omega^2 \cancel{e^{i\omega t}} = -\frac{K}{m}\cancel{Ae^{i\omega t}}$$

$$\omega = \sqrt{\frac{K}{m}}$$

Book asks them to memorize, do problem in your head. I need to tell them that it is not needed to memorize anything or do anything in one's head.