Math Methods in Physics I

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class #22

Fourier Series

Power series don't do well with periodic or discontinuous functions.

Let's use periodic functions instead

$$f(x) = C + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

Integrate over a full period

$$\int_{-\pi}^{\pi} f(x)\,dx = C \int_{-\pi}^{\pi} dx + a_1 \int_{-\pi}^{\pi} \cos x \, dx + \dots + b_1 \int_{-\pi}^{\pi} \sin x \, dx + \dots$$

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi C \quad \therefore \quad \boxed{C = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx}$$

For a_1, multiply by $\cos x$ and integrate

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = a_0 \int_{-\pi}^{\pi} \cos(x)\,dx + a_1 \int_{-\pi}^{\pi} \cos^2(x)\,dx + a_2 \int_{-\pi}^{\pi} \cos 2x \cos x \, dx + \dots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin(x)\cos(x)\,dx + b_2 \int_{-\pi}^{\pi} \sin 2x \cos x \, dx + \dots$$

So,

$$\int_{-\pi}^{\pi} \cos 2x \cos x \, dx = \int_{-\pi}^{\pi} \left(\frac{e^{i2x} + e^{-i2x}}{2}\right)\left(\frac{e^{ix} + e^{-ix}}{2}\right) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{i2x} + e^{-i2x}}{2}\right)\left(\frac{e^{ix} + e^{-ix}}{2}\right) dx$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} \left(e^{i3x} + e^{ix} + e^{-ix} + e^{-i3x}\right) dx$$

$$= \frac{1}{4}\left(\int_{-\pi}^{\pi} e^{i3x} + e^{-i3x}\, dx + \int_{-\pi}^{\pi} e^{ix} + e^{-ix}\, dx\right)$$

$$= 2\left[\int_{-\pi}^{\pi} \cos 3x\, dx + \int_{-\pi}^{\pi} \cos x\, dx\right] = 0$$

In general, the integral of <u>any</u> e^{ikx} term over a full period is zero

$$\int_{-\pi}^{\pi} e^{ikx} dx = \frac{e^{ikx}}{ik} = \frac{e^{ik\pi} - e^{-ik\pi}}{ik} = 0 \qquad (\sin k\pi = 0)$$

unless $k=0$, that integral zeroes. So,

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} e^{imx} dx + \int_{-\pi}^{\pi} e^{-imx} dx + \int_{-\pi}^{\pi} e^{inx} dx + \int_{-\pi}^{\pi} e^{-inx} dx$$

$= 0$ if $m \neq n$

$n\pi = u$

what if $m=n$?

$$\int_{-\pi}^{\pi} (\cos nx)^2 dx = \frac{1}{n}\int_{-n\pi}^{n\pi} \cos^2 u\, du = \frac{1}{2n}\left(nx + \sin nx \cos nx\right)\Big|_{-\pi}^{\pi}$$

$nx = u \qquad n dx = du$

$$= \frac{\cancel{2}\pi}{\cancel{2}}$$

If $m=n=0$?

$$\int_{-\pi}^{\pi} (\cos nx)^2 \, dx = \int_{-\pi}^{\pi} dx = 2\pi$$

So
$$\boxed{\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases}}$$

Back to:

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = a_0 \int_{-\pi}^{\pi} \cos(x)\,dx + a_1 \int_{-\pi}^{\pi} \cos^2(x)\,dx + a_2 \int_{-\pi}^{\pi} \cos 2x \cos x \, dx + \dots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin(x)\cos(x)\,dx + b_2 \int_{-\pi}^{\pi} \sin 2x \cos x \, dx + \dots$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\cos x \, dx = a_1 \cdot \pi \qquad \boxed{a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(x)\,dx}$$

For a_2, mutiply by $\cos 2x$

$$\int_{\pi}^{\pi} f(x)\cos 2x\,dx = a_0\int_{\pi}^{\pi}\cos 2x\,dx \cdot a_1\int_{\pi}^{\pi}\cos x\cos 2x\,dx + a_2\int_{-\pi}^{\pi}(\cos 2x)^2\,dx +$$
$$+\ b_1\int_{-\pi}^{\pi}\sin x\cos 2x\,dx + \dots$$

$$\int_{\pi}^{\pi} f(x)\cos 2x\,dx = a_2\cdot\pi \qquad \therefore \boxed{a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

As for b_n? Multiply by sines!

$$\int_{\pi}^{\pi} f(x)\sin(x)\,dx = a_0\cancel{\int_{-\pi}^{\pi}\sin(x)\,dx} + a_1\cancel{\int_{-\pi}^{\pi}\cos x\sin x\,dx} + \dots$$
$$+\,b_1\int_{\pi}^{\pi}\sin^2 x\,dx + b_2\int_{-\pi}^{\pi}\sin x\sin 2x\,dx$$

$$\int_{-\pi}^{\pi}\sin^2 nx\,dx = \left[\frac{x}{2} - \frac{\sin(2nx)}{4n}\right]\Bigg|_{-\pi}^{\pi} = \pi \qquad (m=n\neq 0)$$

$$\int_{\pi}^{\pi}\sin^2 nx = 0 \quad \text{for } m=n=0$$

$$\int_{-\pi}^{\pi}\sin nx\sin mx\,dx = \int e^{inx} - \int e^{-inx} + \int e^{imx} - \int e^{-imx} = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ 1/2, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} f(x) \sin(x) \, dx = b_1 \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(x) \, dx$$

b_2:

$$\int_{-\pi}^{\pi} f(x) \sin 2x \, dx = \frac{b}{2} \int_{-\pi}^{\pi} (\sin 2x)^2 \, dx = \pi b_2$$

$$b_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin 2x \, dx$$

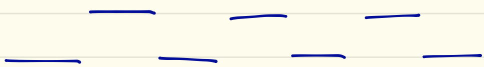
$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx}$$

Write $c = \frac{a_0}{2}$ and group with a's $(\cos 0x = 1, \sin 0x = 0)$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos(nx) + b_n \sin(nx)}$$

Expand in Fourier series the periodic box function

$$f(x) = \begin{cases} 0 & ; \ -\pi < x < 0 \\ 1 & ; \ 0 < x < \pi \end{cases}$$



a and b?

$$a_n = \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos nx \, dx + \int_{0}^{\pi} \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos nx \, dx = \begin{cases} \frac{1}{\pi}\cdot\frac{1}{n} \sin nx \Big|_0^{\pi} = 0 & \text{for } n \neq 0 \\ \frac{1}{\pi}\cdot \pi = 1 & \text{for } n = 0 \end{cases}$$

So, $a_0 = 1$, all other $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \sin nx \, dx + \int_0^{\pi} 1 \cdot \sin nx \, dx \right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin nx \, dx = \frac{1}{\pi}\left[-\frac{\cos nx}{n} \right]\Big|_0^{\pi} = -\frac{1}{n\pi}\left[(-1)^n - 1 \right]$$

$$= \begin{cases} 0 & \text{for even } n \\ \frac{2}{n\pi} & \text{for odd } n \end{cases}$$

So,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right)$$

other functions:

$$g(x) = \begin{cases} -1 , & -\pi < x < 0 \\ 1 , & 0 < x < \pi \end{cases}$$

$$g(x) = 2f(x) - 1$$

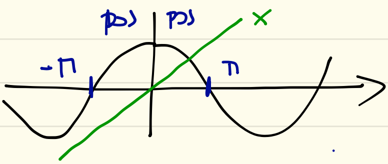
$$= \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

change phase: $h(x) = f(x + \pi/2)$ $\Rightarrow$ figure shifted to the left

$$h(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{3}\right)$$

$f(x) = x \; ; \; -\pi < x < \pi$

$$a_n = \int_{-\pi}^{\pi} x \cos nx \, dx = 0$$ (general result for any odd function $f(x) = -f(-x)$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{1}{\pi}\left[\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$b_n = -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - , \ldots\right]$$