Math Methods in Physics I

Prof Wladimir Lyra

Sep 8, 2016

Class #4

Accuracy of Series approximations:

Remainder:

Given $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$

$$\boxed{R_n(x) = f(x) - \sum_{k=0}^{n} f^{(k)}(0) \frac{x^k}{k!}}$$

Terms of the series of order higher than n.

Converges if $\lim_{n\to\infty} |R_n(x)| = 0$

$f(x) = T_n(x) + R$

Theorem

$$\boxed{R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt}$$

Proof: By induction

For $n=1$

$$R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x-a)$$

Consider

$$\int_a^x (x-t) f''(t)\,dt$$

Use integration by parts

$$\int u\,dv = uv - \int v\,du \quad \text{with} \quad u = x-t \;;\; du = -dt$$
$$dv = f''(t)\,dt \;;\; v = f'(t)$$

Then

$$\int_a^x (x-t) f''(t)\,dt = (x-t) f'(t) \Big|_{t=a}^{t=x} + \int_a^x f'(t)\,dt$$

$$= \cancel{0 f'(x)} - (x-a) f'(a) + f(x) - f(a)$$

$$f(x) - \underbrace{\left[f(a) + (x-a) f'(a)\right]}_{T_1(x)} = R_1(x) = \int_a^x (x-t) f''(t)\,dt$$

So, the theorem is proved for $n=1$.

Suppose now that it is true for $n=k$.

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$$

Let's show that if this true for $n=k$, then it is true for $n=k+1$

$$R_{k+1}(x) = \frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,dt$$

Again use integration by parts

$$\int u\,dv = uv - v\int du$$

with

$$u = (x-t)^{k+1} \qquad dv = f^{(k+2)}(t)$$

$$du = -(k+1)(x-t)^k dt \qquad v = f^{(k+1)}(t)$$

So $$\frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,dt =$$

$$= \frac{1}{(k+1)!}(x-t)^{k+1} f^{(k+1)}(t)\Big|_{t=a}^{t=x} + \frac{k+1}{(k+1)!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$$

$$= 0 - \frac{1}{(k+1)!}(x-a)^{k+1} f^{(k+1)}(a) + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$$

$$= -\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + R_k(x)$$

$$= R_k(x) - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}$$

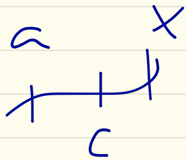
$$= f(x) - T_k(x) - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}$$

$$= f(x) - \left(T_k(x) + \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}\right)$$

$$= f(x) - T_{k+1}(x) = R_{(k+1)}(x)$$

Proved that it is true for $n=1$ and that, if true for $n=k$, it will be true for $n=k+1$. So, the theorem is proved by induction.

A more useful form of the theorem:



$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

Making the interval as small as possible

The derivative approaches a constant value at point c

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!} \int_a^x (x-t)^n \, dt$$

$$= -\frac{f^{(n+1)}(c)}{n!} \cdot \frac{(x-t)^{n+1}}{n+1} \Big|_a^x$$

$$= f^{(n+1)}(c) \cdot \frac{(x-a)^{n+1}}{(n+1)!}$$

$$\therefore \quad \boxed{R_n(x) = f^{(n+1)}(c) \cdot \frac{(x-a)^{n+1}}{(n+1)!}}$$

Example: $f(x) = \sin(x) = 1 + x + x^3/3! + x^5/5! + \ldots$

$$R_n(x) = f^{(n+1)}(c) \, \frac{x^{(n+1)}}{(n+1)!}$$

c indeterminate, but

$$f^{(n+1)}(c) = \pm \sin(c) \text{ or } \pm \cos(c) \quad \leq 1$$

$$\text{so } R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$

And as $n \to \infty$, $R_n(x) \to 0$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

goes to zero for any x, so the Taylor series of the sin is true for any x.

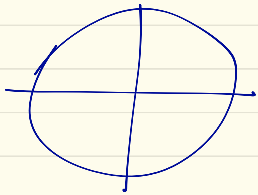
$$\frac{1}{n^2} < \frac{1}{n(n-1)}$$

$$\frac{1}{n(n-1)} = \frac{n-(n-1)}{n(n-1)} = \frac{n}{n(n-1)} - \frac{(n-1)}{n(n-1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots - \frac{1}{n}$$

$$= 1$$

Complex numbers



Polar form:

$$z = x + iy$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

Useful for multiplying or dividing complex numbers

$$z_1 = r_1 e^{i\theta_1} \qquad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$z_2 = r_2 e^{i\theta_2} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$z^n = r^n e^{in\theta}$

for $r=1$

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$\sin 2\theta$?

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$

$$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta) = \cos 2\theta + i\sin 2\theta$$

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

Example

$$\left[\cos\frac{\pi}{10} + i\sin\left(\frac{\pi}{10}\right)\right]^{25} = \left(e^{i\frac{\pi}{10}}\right)^{25} = e^{i\frac{25}{10}\pi}$$

$$= e^{i\frac{5}{2}\pi} = e^{i2\pi} \cdot e^{i\frac{\pi}{2}} = 1 \cdot i = i$$ Circle 25 times!

Trick: use $\tau = 2\pi$ instead of π

$\tau = 2\pi$

$\frac{\pi}{10} \to \frac{\tau}{20}$

$$\left(e^{i\tau/20}\right)^{25} = e^{i\frac{25}{20}\tau} = e^{i\frac{5\tau}{4}} = e^{i\tau} \cdot e^{i\tau/4}$$

$\boxed{e^{i\tau} = 1}$ $e^{i\tau}$ full turn

— // —

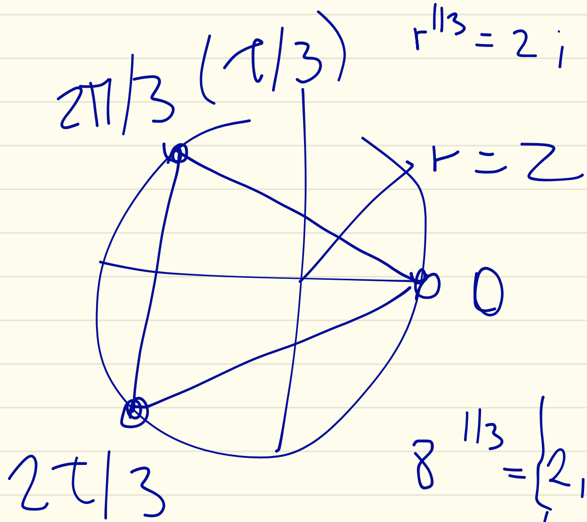
Roots

Find roots of $z^3 = 8$

$z^{1/3} = r^{1/3} e^{i\theta/3}$

$z = 8 \Rightarrow z = re^{i\theta}$, $r = 8$; $\theta = 2k\pi = 0, 2\pi, 4\pi, \ldots$

$r^{1/3} = 2$; $\theta = \frac{2k\pi}{3} = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

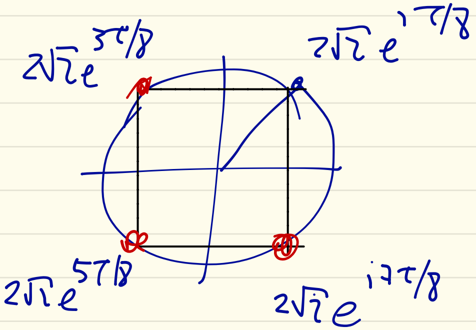


roots in a triangle

$$8^{1/3} = \left\{2, 2e^{i2\pi/3}, 2e^{i4\pi/3}\right\}$$
$$= \left\{2, -1 + i\sqrt{3}, -1 - i\sqrt{3}\right\}$$

$\sqrt[4]{-64}$: $z = -64 = re^{i\theta}$ $r = 64$; $\theta = \pi + 2k\pi$

$z^{1/4} = r^{1/4} e^{i\theta/4} = 2\sqrt{2}\, e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)}$; $k = 0, 1, 2$



$2\sqrt{2}\left\{ e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4} \right\}$

$\sqrt[4]{64} = \pm 2 \pm 2i$