Math Methods in Physics I

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Class #19

start with extra credit? or end with it? Two problems.

$f = \cos z = \cos(x+iy)$; We know $\frac{df}{dz} = -\sin z$, but what is $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$?

$y = f(x)$ (Line in plane)

$z = f(x,y)$ (curve in 3D space)

Keep x constant. z is a function of y. We could write $\frac{dz}{dy}$, but as z is also a function of x, we write instead

$$\frac{\partial z}{\partial y}$$

and call $\frac{\partial z}{\partial y}$ the partial derivative of z with respect to y.

"partial dee", "partial", curly ∂

Lowercase cyrillic ∂.

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$\frac{\partial}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2}$; $\frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y}$; $\frac{\partial}{\partial x}\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^3 z}{\partial x^2 \partial y}$

$z = f(x,y) = x^3 y - e^{xy}$

Calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ $\frac{\partial^2 f}{\partial x^2}$ $\frac{\partial^3 f}{\partial y^3}$ $\frac{\partial^3 f}{\partial x^2 \partial y}$

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$z = x^2 - y^2$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\qquad \left(\frac{\partial z}{\partial r}\right)_\theta = 2r(\cos^2\theta - \sin^2\theta)$ equiv $z = f(r,\theta)$

$z = 2x^2 - x^2 - y^2 = 2x^2 - r^2$ $\qquad \left(\frac{\partial z}{\partial r}\right)_x = -2r$ $\qquad g(r,x)$

$z = x^2 + y^2 - 2y^2 = r^2 - 2y^2$ $\qquad \left(\frac{\partial z}{\partial r}\right)_y = 2r$ $\qquad h(r,y)$

These are basically 3 different functions.

$z = f(r,\theta) = g(r,x) = h(r,y)$

$\left(\frac{\partial z}{\partial r}\right)_x = \frac{\partial g}{\partial r}$ $\qquad \left(\frac{\partial z}{\partial r}\right)_y = \frac{\partial h}{\partial r}$

Inconvenient when letters hv physical meaning. In thermo

$\left(\frac{\partial T}{\partial p}\right)_V$; $\left(\frac{\partial T}{\partial V}\right)_S$; $\left(\frac{\partial T}{\partial p}\right)_u$; $\left(\frac{\partial T}{\partial S}\right)_p$, etc

Series expansion

$f = \cos z = \cos(x+iy)$; We know $\frac{df}{dz} = -\sin z$, but what is $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$?

Series expansion about point (a,b). What is the surface that approximates the function?

$f(x,y) = e^x \log(1+y)$ 2^{nd} order about $(a,b)=(0,0)$

$f_x = e^x \log(1+y)$ $f_x(0,0) = 0$

$f_y = e^x (1+y)^{-1}$ A-7 001161N $f_y(0,0) = 1$

$f_{xx} = e^x \log(1+y)$ $f_{xx}(0,0) = 0$

$f_{yy} = -e^x (1+y)^{-2}$ $f_{yy}(0,0) = -1$

$f_{xy} = f_{yx} = e^x (1+y)^{-1}$ $f_{xy}(0,0) = f_{yx}(0,0) = 1$

$$T = y + xy - \frac{y^2}{2} + \dots$$

$f(x,y)$ around (a,b) $\Rightarrow$ write the powers in terms of $(x-a)$ and $(y-b)$

$$f(x,y)\Big|_{(a,b)} = \left(\sum_n c_n (x-a)^n\right)\left(\sum_k d_k (y-b)^k\right)$$

$$f(x,y) = a_{00} + a_{10}(x-a) + a_{01}(y-b) + a_{20}(x-a)^2 + a_{11}(x-a)(y-b)$$
$$+ a_{02}(y-b)^2 + a_{30}(x-a)^3 + a_{21}(x-a)^2(y-b)$$
$$+ a_{12}(x-a)(y-b)^2 + a_{03}(y-b)^3 + \ldots$$

Take the derivatives

$$f_x = a_{10} + 2a_{20}(x-a) + a_{11}(y-b) + \ldots$$
$$f_y = a_{01} + a_{11}(x-a) + 2a_{02}(y-b) + \ldots$$
$$f_{xx} = 2a_{20} + \ldots$$
$$f_{xy} = a_{11} + \ldots$$

put $x=a$; $y=b$

$f(a,b) = a_{00}$ $\quad f_x(a,b) = a_{10}$ $\quad f_y(a,b) = a_{01}$

$f_{xx}(a,b) = 2a_{20}$ $\quad f_{xy}(a,b) = a_{11}$

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \right] + \dots$$

write $h = (x-a)$ $\quad K = (y-b)$

2^{nd} order:

$$\frac{1}{2!}\left[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)K^2 \right]$$
$$= \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f(a,b)$$

The 3^{rd} order term is

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^3 f(a,b)$$

$$\boxed{f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right] f(a,b)}$$

$\boxed{f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ is the tangent plane.

"Total" derivative

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$\Delta z = f(x+\Delta x, y) - f(x,y) + f(x+\Delta x, y+\Delta x) - f(x+\Delta x, y)$$

Given $f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$\Delta z = \frac{\partial f}{\partial x}(x,y)\Delta x + \frac{\partial f}{\partial y}(x,y)\Delta y$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Say $z = f(x,y,t)$
then, dividing by t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Variation of z along tangent plane

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u_x + \frac{\partial u}{\partial y}u_y + \frac{\partial u}{\partial z}u_z$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u$$

$$\frac{\partial}{\partial t} + (u \cdot \nabla) u$$

$$\frac{du}{dt} = \left(\frac{\partial}{\partial t} + u \cdot \nabla \right) u$$