INNER BINARY AVERAGE

$$\langle H_{PERT} \rangle \approx -\frac{G m_1}{4} \frac{r^2}{r_1^3} \left[1 - 3\sin^2(\omega+f)\sin^2 I \right]$$

θ	J
M	E
ω	L
Ω	L_z

$$\frac{\partial \langle\langle H_{PERT} \rangle\rangle}{\partial M} = \dot{E} = 0$$

$$a \equiv \text{const}$$

$$\boxed{(1-e^2)^{1/2} \cos I \equiv \text{const}}$$

$$\frac{r^2}{R_1^3} = \frac{r^2}{a^2} \cdot \frac{a^2}{a_1^3}$$

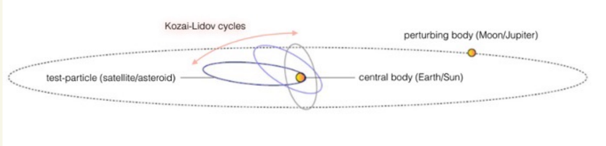
$$\left(\frac{r}{a}\right)^2 \approx 1 - 2e\cos M + \frac{e^2}{2}\left(3 - \cos 2M\right)$$

$$\sin^2(\omega+f) = \left[\sin\omega\cos f + \sin f\cos\omega\right]^2$$

$$= \sin^2\omega\cos^2 f + \sin^2 f\cos^2\omega + 2\sin\omega\cos\omega\sin f\cos f$$

$$\boxed{\langle\langle H_{PERT} \rangle\rangle = -\frac{G m_1}{8} \frac{a^2}{a_1^3} \left[2 + 3e^2 - 3\left(1 - e^2 + 5e^2\sin^2\omega\right)\sin^2 I \right]}$$

OSCILLATIONS



$$\langle\langle H_{PERT}\rangle\rangle = -\frac{Gm_1}{8}\frac{a^2}{a_1^3}\left[2+3e^2-3(1-e^2+5e^2\sin^2\omega)\sin^2 I\right]$$

$$\dot{J}_i = -\frac{\partial H}{\partial \theta_i}$$

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i}$$

ANGLE θ	ACTION J
$\theta_1 = M$	$J_1 = (\mu a)^{1/2}$
$\theta_2 = \omega$	$J_2 = L = J_1(1-e^2)^{1/2}$
$\theta_3 = \Omega$	$J_3 = L_z = J_2 \cos I$

$$\dot{J}_1 = \frac{\partial H}{\partial M} = 0 \quad \therefore \quad \dot{a} = 0$$

$$\dot{J}_3 = \frac{\partial H}{\partial \Omega} = 0 \quad \therefore \quad \dot{L}_z = 0$$

$$\dot{J}_2 = -\frac{\partial H}{\partial \omega} = \dot{L} \neq 0$$

$$= -\frac{15}{8} Gm_1 \frac{a^2}{a_1^3} e^2 \sin^2 I \sin 2\omega$$

$$L = [\mu a(1-e^2)]^{1/2}$$

$$(\mu \cancel{a})^{1/2} \cdot \frac{(-\cancel{e}\dot{e})}{(1-e^2)^{1/2}} = -\frac{15}{8} Gm_1 \frac{a^{\cancel{2}\,3/2}}{a_1^3} e^{\cancel{2}} \sin^2 I \sin 2\omega$$

$$\mu = G(m_0+m_1)$$

$$(\cancel{\mu a})^{1/2} \cdot \frac{(-\cancel{e}\dot{e})}{(1-e^2)^{1/2}} = -\frac{15}{8} G m_1 \frac{a^{\cancel{2}\,3/2}}{a_1^3} e^{\cancel{2}} \sin^2 I \sin 2\omega \qquad \mu = G(m_0+m_1)$$

$$\dot{e} = \frac{15}{8} \overset{=n_1^2}{\left(\frac{G m_1}{a_1^3}\right)} \underset{=n}{\left[\frac{G(m_0+m_1)}{a^3}\right]^{-1/2}} e\,(1-e^2)^{1/2} \sin^2 I \sin 2\omega$$

$$\boxed{\dot{e} = \frac{15}{8}\frac{n_1^2}{n} e\,(1-e^2)^{1/2} \sin^2 I \sin 2\omega}$$

$$L_z = [\mu a(1-e^2)]^{1/2} \cos I = L \cos I$$

$$\dot{L}_z = \dot{L} \cos I - L \sin I\, \dot{I} = 0$$

$$= (\cancel{\mu a})^{1/2} \cdot \frac{(-e\dot{e})}{(1-e^2)^{1/2}} \cos I - (\cancel{\mu a})^{1/2} (1-e^2)^{1/2} \sin I\, \dot{I} = 0$$

$$\dot{I} = -\frac{e}{(1-e^2)} \frac{\cos I}{\sin I} \dot{e} = -\frac{e^2}{(1-e^2)^{1/2}} \frac{\cos I}{\cancel{\sin I}} \frac{15}{8} \frac{n_1^2}{n} \cancel{e}\, \cancel{(1-e^2)^{1/2}} \sin^{\cancel{2}} I \sin 2\omega$$

$$\boxed{\dot{I} = -\frac{15}{16}\frac{n_1^2}{n} e^2 (1-e)^{-1/2} \sin 2I \sin 2\omega}$$

CRITICAL ANGLE

$$\langle\langle H_{PERT} \rangle\rangle = -\frac{Gm_1}{8}\frac{a^2}{a_1^3}\left[2+3e^2-3\left(1-e^2+5e^2\sin^2\omega\right)\sin^2 I\right]$$

$$\dot{J}_i = -\frac{\partial H}{\partial \theta_i}$$

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i}$$

ANGLE θ	ACTION J
$\theta_1 = M$	$J_1 = (\mu a)^{1/2}$
$\theta_2 = \omega$	$J_2 = L = J_1(1-e^2)^{1/2}$
$\theta_3 = \Omega$	$J_3 = L_z = J_2 \cos I$

$$\dot{\omega} = \frac{\partial H}{\partial J_2}$$

$J_1 = (\mu a)^{1/2} \quad \therefore$

$J_2 = J_1(1-e^2)^{1/2} \quad \therefore$

$J_3 = J_2 \cdot \cos I \quad \therefore$

$$\boxed{\begin{aligned} a &= J_1^2/\mu \\ e^2 &= 1-\frac{J_2^2}{J_1^2} \\ \sin^2 I &= 1-\frac{J_3^2}{J_2^2} \end{aligned}}$$

KOZAI-LIDOV HAMILTONIAN IN ACTIONS:

$$\langle\langle H_{PERT} \rangle\rangle = -\frac{Gm_1 a^2}{8a_1^3}\left[5+3\frac{J_3^2}{J_1^2}-6\frac{J_2^2}{J_1^2}-15\left(1-\frac{J_2^2}{J_1^2}-\frac{J_3^2}{J_2^2}+\frac{J_3^2}{J_1^2}\right)\sin^2\omega\right]$$

$$\dot{\omega} = \frac{\partial H}{\partial J_2} = \frac{3Gm_1a^2}{4a_1^3 J_2}\left[2\frac{J_2^2}{J_1^2}+5\left(\frac{J_3^2}{J_2^2}-\frac{J_2^2}{J_1^2}\right)\sin^2\omega\right]$$

$$\langle\langle H_{PERT} \rangle\rangle = -\frac{G m_1 a^2}{8 a_1^3}\left[5 + 3\frac{J_3^2}{J_1^2} - 6\frac{J_2^2}{J_1^2} - 15\left(1 - \frac{J_2^2}{J_1^2} - \frac{J_3^2}{J_2^2} + \frac{J_3^2}{J_1^2}\right)\sin^2\omega\right]$$

$$\dot{\omega} = \frac{\partial H}{\partial J_2} = \frac{3 G m_1 a^2}{4 a_1^3 J_2}\left[2\frac{J_2^2}{J_1^2} + 5\left(\frac{J_3^2}{J_2^2} - \frac{J_2^2}{J_1^2}\right)\sin^2\omega\right]$$

For $\dot{\omega} = 0$ $\quad \sin^2\omega = 1 \therefore \omega = \pm 90^\circ$

$$2\frac{J_2^2}{J_1^2} + 5\left(\frac{J_3^2}{J_2^2} - \frac{J_2^2}{J_1^2}\right) = 0$$

$$-3\frac{J_2^2}{J_1^2} + 5\frac{J_3^2}{J_2^2} = 0 \quad \therefore \quad J_2 = \left[\frac{5}{3} J_3^2 J_1^2\right]^{1/4}$$

$$\boxed{J_1 = (\mu a)^{1/2} \qquad J_2 = J_1(1-e^2)^{1/2} \qquad J_3 = J_2 \cdot \cos I}$$

$$J_2 = \left[\frac{5}{3} J_2^2 \cos^2 I \, J_1^2\right]^{1/4} = J_2^{1/2}\left[\frac{5}{3}\cos^2 I \, J_1^2\right]^{1/4}$$

$$J_2 = \left[\frac{5}{3}\cos^2 I \, J_1^2\right]^{1/2} = \left(\frac{5}{3}\right)^{1/2} J_1 \cos I$$

$$(1-e^2)^{1/2} = \left(\frac{5}{3}\right)^{1/2}\cos I \qquad \text{Set } e \to 0$$

$$\boxed{\cos I_{CRIT} = \left(\frac{3}{5}\right)^{1/2} \approx 40^\circ}$$