KEPLER PROBLEM IN ANGLE-ACTION VARIABLES

$$\ell = M \qquad L = (\mu a)^{1/2} = E \text{ (ENERGY)}$$

$$g = \omega \qquad G = L(1-e^2)^{1/2} = h \text{ (ANGULAR MOMENTUM)}$$

$$h = \Omega \qquad H = G\cos I = h_z \text{ (VERTICAL COMPONENT OF ANGULAR MOMENTUM)}$$

HAMILTONIAN

$$H = \frac{p^2}{2} + \Phi(r) \qquad p = \nabla S$$

$$= \frac{|\nabla S|^2}{2} + \Phi(r)$$

IN SPHERICAL COORDINATES

$$H = \frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\right)^2\right] + \Phi(r)$$

$$S \equiv S_r(r) + S_\theta(\theta) + S_\phi(\phi)$$

$$p_i = \frac{\partial S}{\partial q_i} \qquad p_r = \frac{\partial S_r}{\partial r}\ ; \quad p_\theta = \frac{\partial S_\theta}{\partial \theta}\ ; \quad p_\phi = \frac{\partial S_\phi}{\partial \phi}$$

$$p_\phi = \frac{\partial S_\phi}{\partial \phi} = L_z$$

$$H = \frac{1}{2}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S_\theta}{\partial \theta}\right)^2 + \frac{L_z^2}{r^2 \sin^2\theta} \right] + \Phi(r)$$

$$p_\theta^2 = \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = L^2 + f$$

$$H = \frac{1}{2}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{L^2}{r^2} + \cancel{\frac{f}{r^2}} + \cancel{\frac{L_z^2}{r^2 \sin^2\theta}} \right] + \Phi(r)$$

$$f = -\frac{L_z^2}{\sin^2\theta}$$

$$p_\theta^2 = L^2 - \frac{L_z^2}{\sin^2\theta}$$

$$H = \frac{1}{2}\left[p_r^2 + p_\theta^2 + p_\phi^2 \right] + \Phi(r) = E$$

$$\frac{1}{2}\left[p_r^2 + \frac{L^2}{r^2} \right] + \Phi(r) = E$$

$$p_r^2 = \left(\frac{\partial S_r}{\partial r}\right)^2 = 2E - 2\Phi(r) - \frac{L^2}{r^2}$$

$$p_\theta^2 = \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = L^2 - \frac{L_z^2}{\sin^2\theta}$$

$$p_\phi^2 = \left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = L_z^2$$

INTEGRATE THE GENERATING FUNCTION

$$S = \int \frac{\partial S_\phi}{\partial \phi} d\phi + \int \frac{\partial S_\theta}{\partial \theta} d\theta + \int \frac{\partial S_r}{\partial r} dr$$

$$= \int L_z \, d\phi + \int \left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2} d\theta + \int \left[2E - 2\Phi(r) - \frac{L^2}{r^2}\right]^{1/2} dr$$

$$J_i = \frac{1}{2\pi} \oint \frac{\partial S}{\partial q} \cdot dq = \frac{\Delta S}{2\pi}$$

CHANGE ϕ, KEEP θ, r FIXED

$$J_\phi = \frac{\Delta S}{2\pi} = \frac{1}{2\pi} \oint \frac{\partial S}{\partial \phi} d\phi = \frac{1}{\cancel{2\pi}} \cdot L_z \oint \cancel{d\phi} = L_z$$

$$J_\phi = L_z$$

FIRST ACTION FOUND (AZIMUTHAL ACTION)

$$J_i = \frac{1}{2\pi} \oint \frac{\partial S}{\partial q} \cdot dq$$

CHANGE θ, KEEP ϕ, r FIXED

$$J_\theta = \frac{1}{2\pi} \int \frac{\partial S}{\partial \theta} d\theta = \frac{1}{2\pi} \int \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\, d\theta$$

$$= L - L_z$$

CHANGE r, KEEP ϕ, θ FIXED

$$J_r = \frac{1}{\pi} \int_{r_{MIN}}^{r_{MAX}} \left[2E - 2\Phi(r) - \frac{L^2}{r^2}\right]^{1/2} dr \qquad \Phi = -\frac{GM}{r}$$

$$= \frac{1}{\pi} \int_{r_{MIN}}^{r_{MAX}} \left[2E + \frac{2GM}{r} - \frac{L^2}{r^2}\right]^{1/2} dr = \frac{GM}{\sqrt{-2E}} - L$$

$$J_r = \frac{GM}{\sqrt{-2E}} - L \quad ; \quad J_\theta = L - L_z \quad ; \quad J_\phi = L_z$$

$$L = J_\theta + J_\phi$$

$$J_r + J_\theta + J_\phi = \frac{GM}{\sqrt{-2E}} \rightarrow E = -\frac{1}{2}\left(\frac{GM}{J_r + J_\theta + J_\phi}\right)^2$$

$$H = -\frac{1}{2}\left(\frac{GM}{J_r + J_\theta + J_\phi}\right)^2$$

FINDING THE ANGLES

$$H = -\frac{1}{2}\left(\frac{GM}{J_r + J_\theta + J_\phi}\right)^2$$

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} \qquad \Omega_r = \Omega_\theta = \Omega_\phi = \frac{(GM)^2}{(J_r + J_\theta + J_\phi)^3}$$

SET $J_r + J_\theta + J_\phi$ AS AN ACTION

$[J_a, J_b, J_c]$

$J_b = J_\theta + J_\phi = L$

$J_\theta = L - L_z \; ; \; J_\phi = L_z$

$J_b = L \; ; \; J_a = L_z \; ; \; J_c = J_r + L$

$$\boxed{\begin{aligned} J_\phi &= J_a \\ J_\theta &= J_b - J_a \\ J_r &= J_c - J_b \end{aligned}} \iff \boxed{\begin{aligned} J_a &= L_z \\ J_b &= L \\ J_c &= J_r + L \end{aligned}} \qquad H = -\frac{1}{2}\left(\frac{GM}{J_c}\right)^2$$

$$\boxed{\begin{aligned} J_a &= L_z \\ J_b &= L \\ J_c &= J_r + L \end{aligned}}$$

$$H = -\frac{1}{2}\frac{(GM)^2}{J_c^2}$$

$$\dot{\theta}_i = \Omega_i = \frac{\partial H}{\partial J_i}$$

$$\Omega_a = \Omega_b = 0$$

$$\dot{\theta}_a = \dot{\theta}_b = 0$$

$$\Omega_c = \frac{(GM)^2}{J_c^3}$$

θ_c PERIODIC

$$\theta_i = \frac{\partial S}{\partial J_i} \qquad J_a = L_z \, ; \quad J_b = L$$

$$S = \int L_z \, d\phi + \int \left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2} d\theta + \int \left[2E - 2\Phi(r) - \frac{L^2}{r^2}\right]^{1/2} dr$$

$$= \phi J_a + \int \left(J_b^2 - \frac{J_a^2}{\sin^2\theta}\right)^{1/2} d\theta + \int \left[2E - 2\Phi(r) - \frac{J_b^2}{r^2}\right] dr$$

$$\theta_a = \frac{\partial S}{\partial J_a}$$

$$= \phi + \frac{\partial}{\partial J_a} \int \left(J_b^2 - \frac{J_a^2}{\sin^2\theta}\right)^{1/2} d\theta = \phi - u$$

$$u = -\frac{\partial}{\partial J_a} \int \left(J_b^2 - \frac{J_a^2}{\sin^2\theta}\right)^{1/2} d\theta$$

$$= \int \frac{1}{\cancel{2}\sqrt{J_b^2 - \frac{J_a^2}{\sin^2\theta}}} \cdot \left(\frac{\cancel{2} J_a}{\sin^2\theta}\right) d\theta$$

$$= J_a \int \frac{1}{\sin^2\theta} \frac{1}{\sqrt{J_b^2 - J_a^2/\sin^2\theta}} \, d\theta$$

$$L_z = L \cdot \cos I \quad \therefore \quad \frac{L_z}{L} = \frac{J_a}{J_b} = \cos I$$

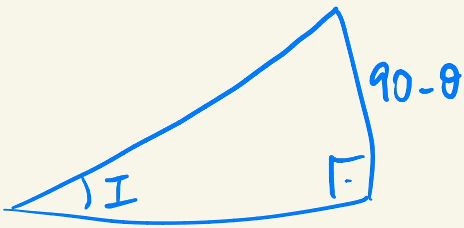
$$u = J_a \int \frac{1}{\sin^2\theta} \frac{1}{\sqrt{J_b^2 - J_a^2/\sin^2\theta}} \, d\theta$$

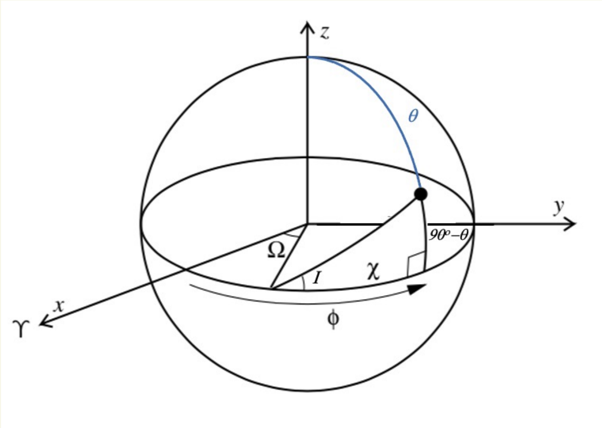
$$= \cancel{J_a} \cdot \int \frac{1}{\sin^2\theta} \frac{1}{\cancel{J_a}\sqrt{\frac{1}{\cos^2 I} - \frac{1}{\sin^2\theta}}} \, d\theta$$

$$= \int \frac{1}{\sin^2\theta} \frac{\cancel{\sin\theta}}{\sqrt{\sec^2 I \sin^2\theta - 1}} \, d\theta$$

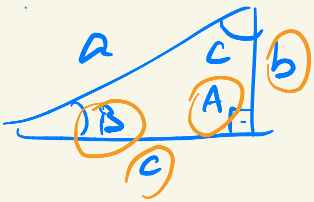
$$u = \int \frac{1}{\sin\theta} \frac{1}{\sqrt{\sec^2 I \sin^2\theta - 1}} \, d\theta$$

$$u = \int \frac{1}{\sin\theta} \frac{1}{\sqrt{\sec^2 I \sin^2\theta - 1}} \, d\theta$$





$\chi = \phi - \Omega$



$$\cot b \sin c = \cot B \sin A + \cos c \cos A$$
$$\cot b \sin a = \cot B \sin C + \cos a \cos C$$
$$\cot a \sin c = \cot A \sin B + \cos c \cos B$$
$$\cot a \sin b = \cot A \sin C + \cos b \cos C$$
$$\cot c \sin a = \cot C \sin B + \cos a \cos B$$
$$\cot c \sin b = \cot C \sin A + \cos b \cos A$$

c, b   B A

$$\cot(90-\theta) \sin\chi = \cot I$$

$$\sin\chi = \cot I \cot\theta$$

$$u = \int \frac{1}{\sin\theta} \frac{1}{\sqrt{\sec^2 1 \sin^2\theta - 1}} \, d\theta$$

$$\sin x = \cot 1 \cot\theta$$

$$\sec^2 = 1 + tg^2$$

$$\csc^2 = 1 + \cot^2$$

$$\sec^2 1 \sin^2\theta - 1 = (1 + tg^2 1)\sin^2\theta - 1$$

$$= \left(1 + \frac{1}{\cot^2 1}\right)\sin^2\theta - 1$$

$$= \left(1 + \frac{1}{\cot^2 1}\right)\left(\frac{1}{1+\cot^2\theta}\right) - 1$$

$$- \sin^2 x$$

$$= \frac{(\cot^2 1 + 1)}{\cot^2 1 (1 + \cot^2\theta)} - 1 = \frac{\cancel{\cot^2 1} + 1 - \cancel{\cot^2 1} - \cot^2 1 \cot^2\theta}{\cot^2 1 (1 + \cot^2\theta)}$$

$$= \frac{\cos^2 x}{\cot^2 1} \cdot \sin^2\theta$$

$$\sin x = \cot 1 \cot\theta$$

$$\cos x \, dx = \frac{\cot 1}{\sin^2\theta} \, d\theta$$

$$d\theta = \frac{\sin^2\theta \cancel{\cos x}}{\cancel{\cot 1}} \, dx$$

$$u = \int \frac{1}{\cancel{\sin^2\theta}} \cdot \frac{\cot \cancel{1}}{\cancel{\cos x}} \, d\theta$$

$$u = \int dx = x = \phi - \Omega$$

$$\theta_a = \frac{\partial S}{\partial J_a}$$

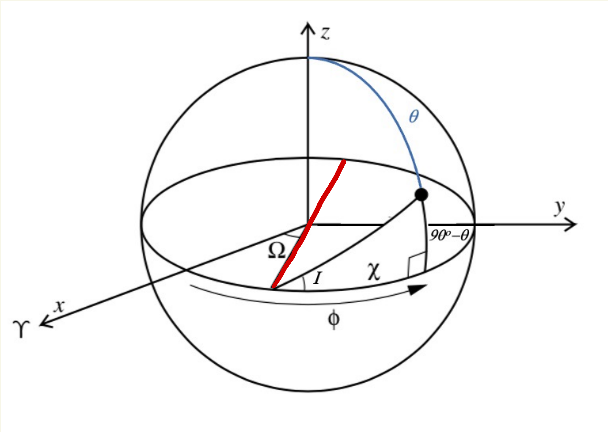
$J_a = L_z$

$u = \phi - \Omega$

$$= \phi + \frac{\partial}{\partial J_a} \int \left(J_b^2 - \frac{J_a^2}{\sin^2\theta} \right)^{1/2} d\theta$$

$$= \phi - u$$

$$\boxed{\theta_a = \Omega \equiv \text{const}}$$



ANGLE	ACTION
Ω	L_z

$\Omega \equiv$ LONGITUDE OF ASCENDING NODE IS FIXED

THE LINE OF NODES IS FIXED

$$J_b = L$$

$$S = \phi J_a + \int \left(J_b^2 - \frac{J_a^2}{\sin^2\theta}\right)^{1/2} d\theta + \int \left[2E - 2\Phi(r) - \frac{J_b^2}{r^2}\right]^{1/2} dr$$

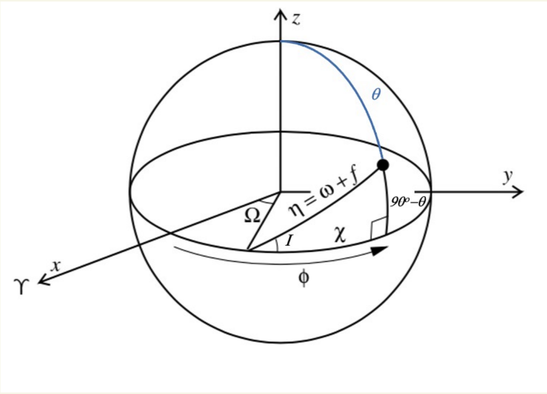
$$\theta_b = \frac{\partial S}{\partial J_b}$$

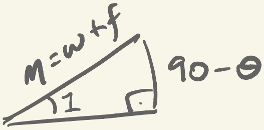
$$= \int \frac{J_b \, d\theta}{\left(J_b^2 - J_a^2/\sin^2\theta\right)^{1/2}} - \int \frac{1}{r^2} \frac{J_b \, dr}{\left[2E - 2\Phi(r) - J_b^2/r^2\right]}$$

$$= C_1 - C_2$$

$$C_1 = \int \frac{J_b \, d\theta}{\left(J_b^2 - J_a^2/\sin^2\theta\right)^{1/2}} = \int \frac{d\theta}{\left(1 - \frac{J_a^2}{J_b^2} \cdot \frac{1}{\sin^2\theta}\right)^{1/2}}$$

$$= \int \frac{d\theta}{\sqrt{1 - \cos^2 I/\sin^2\theta}} = \int \frac{\sin\theta \, d\theta}{\sqrt{\sin^2\theta - \cos^2 I}}$$





$$\frac{\sin(90 - \theta)}{\sin I} = \sin(\omega + f)$$

$$\cos\theta = \sin I \sin\eta$$

$$C_1 = \int \frac{\sin\theta \, d\theta}{\sqrt{\sin^2\theta - \cos^2 I}} \quad ; \quad \cos\theta = \sin I \sin\eta$$

$$-\sin\theta \, d\theta = \sin I \cos\eta \, d\eta$$

$$C_1 = -\int \frac{\sin I \cos\eta \, d\eta}{\sqrt{\sin^2\theta - \cos^2 I}}$$

$$\cos^2\theta = \sin^2 I \sin^2\eta$$

$$1 - \cos^2\theta = 1 - \sin^2 I \sin^2\eta$$

$$= \sin^2\theta$$

$$= -\int \frac{\sin I \cos\eta \, d\eta}{\sqrt{1 - \sin^2 I \sin^2\eta - \cos^2 I}}$$

$$= -\int \frac{\sin I \cos\eta \, d\eta}{\sqrt{\sin^2 I - \sin^2 I \sin^2\eta}} = -\int \frac{\cancel{\cos\eta} \, d\eta}{\sqrt{\cancel{1 - \sin^2\eta}}}$$

$$C_1 = \int d\eta = \eta = \omega + f$$

$$C_1 = \omega + f$$

$$C_2 = \int \frac{1}{r^2} \frac{J_b \, dr}{[2E - 2\Phi(r) - J_b^2/r^2]}$$

$$E = -\frac{\mu}{2a} \quad ; \quad \Phi = -\frac{\mu}{r}$$

$$J_b = L$$

$$= L \int \frac{1}{r^2} \frac{dr}{\left[-\frac{\mu}{a} + \frac{2\mu}{r} - \frac{L^2}{r^2}\right]^{1/2}}$$

$$C_2 = L \int \frac{1}{r^2} \frac{dr}{\left[-\frac{\mu}{a} + \frac{2\mu}{r} - \frac{L^2}{r^2}\right]^{1/2}}$$

$$= \frac{L}{\mu^{1/2}} \int \frac{1}{r^2} \frac{dr}{\left[-\frac{1}{a} + \frac{2}{r} - \frac{L^2}{\mu r^2}\right]^{1/2}}$$

$$= \frac{L}{\mu^{1/2}} \int \frac{1}{r} \frac{dr}{\left[-\frac{r^2}{a} + 2r - \frac{L^2}{\mu}\right]^{1/2}} \qquad L^2 = a\mu(1-e^2)$$

$$= \frac{L}{\mu^{1/2}} \int \frac{1}{r} \frac{dr}{\left[-\frac{r^2}{a} + 2r - a(1-e^2)\right]^{1/2}} \qquad \begin{aligned} r &= a(1-e\cos E) \\ dr &= ae\sin E\, dE \end{aligned}$$

$$= \frac{L}{\mu^{1/2}} \int \frac{\cancel{a} e \sin E\, dE}{\cancel{a}(1-e\cos E)\left[-a(1-e\cos E)^2 + 2a(1-e\cos E) - a(1-e^2)\right]^{1/2}}$$

$$-\cancel{a} + \cancel{2ae\cos E} - \underline{ae^2\cos^2 E} + \cancel{2a} - \cancel{2ae\cos E} - \cancel{a} + \underline{ae^2}$$

$$ae^2(1-\cos^2 E) = ae^2\sin^2 E$$

$$C_2 = \frac{L}{\mu^{1/2}} \int \frac{\cancel{e\sin E}\, dE}{(1-e\cos E)\, a^{1/2}\, \cancel{e\sin E}} = \frac{L}{(a\mu)^{1/2}} \int \frac{dE}{(1-e\cos E)}$$

$$C_2 = \frac{L}{(a\mu)^{1/2}} \int \frac{dE}{(1-e\cos E)}$$

$$\sin f = \frac{\sqrt{1-e^2}\,\sin E}{1-e\cos E}$$

$$\cos f = \frac{\cos E - e}{1-e\cos E}$$

$$\cancel{\cos f}\, df = \sqrt{1-e^2}\,\frac{(\cos E - e)}{(1-e\cos E)^2}\, dE = \frac{\sqrt{1-e^2}\,\cancel{\cos f}\, dE}{(1-e\cos E)}$$

$$dE = \frac{(1-e\cos E)}{\sqrt{1-e^2}}\, df$$

$$L = (a\mu)^{1/2}\sqrt{1-e^2}$$

$$C_2 = \frac{L}{(a\mu)^{1/2}} \cdot \int \frac{(\cancel{1-e\cos E})\, df}{\sqrt{1-e^2}\,(\cancel{1-e\cos E})} = \frac{\cancel{L}}{\cancel{(a\mu)^{1/2}\sqrt{1-e^2}}} \int df$$

$$C_2 = f$$

$$\theta_b = C_1 - C_2 = \omega + \cancel{f} - \cancel{f}$$

$$\boxed{\theta_b = \omega}$$

$$\dot{\theta}_b = \frac{\partial H}{\partial J_b} = 0 \qquad \dot{\omega} = 0$$

ANGLE	ACTION
Ω	L_z
ω	L

J_c ; $H = -\frac{1}{2}\frac{\mu^2}{J_c^2} = E$ $\qquad E = -\frac{\mu}{2a}$

$$J_c^2 = -\frac{\mu^2}{2E} = \mu \cdot a \qquad \boxed{J_c = (\mu a)^{1/2}}$$

$$S = \phi J_a + \int \left(J_b^2 - \frac{J_a^2}{\sin^2\theta}\right)^{1/2} d\theta + \int \left[2E - 2\Phi(r) - \frac{J_b^2}{r^2}\right]^{1/2} dr$$

$$\theta_c = \frac{\partial S}{\partial J_c} = \frac{\partial}{\partial J_c}\int \left[2E - 2\Phi(r) - \frac{J_b^2}{r^2}\right]^{1/2} dr \qquad E = -\frac{\mu^2}{2J_c^2}$$

$$\Phi = -\frac{\mu}{r}$$

$$= \frac{\mu^{1/2}}{a^{3/2}} \int \frac{dr}{\left[-\frac{\mu}{a} + \frac{2\mu}{r} - \frac{L^2}{r^2}\right]^{1/2}}$$

$$= \frac{1}{a^{3/2}} \int \frac{dr}{\left[-\frac{1}{a} + \frac{2}{r} - \frac{L^2}{\mu r^2}\right]^{1/2}}$$

$$= \frac{1}{a^{3/2}} \int \frac{r\,dr}{\left[-\frac{r^2}{a} + 2r - a(1-e^2)\right]^{1/2}} \qquad L^2 = a\mu(1-e^2)$$

$$r = a(1 - e\cos E)$$

$$dr = ae\sin E\, dE$$

$$\theta_c = \frac{1}{a^{3/2}} \int \frac{r\,dr}{\left[-\frac{r^2}{a} + 2r - a(1-e^2)\right]^{1/2}}$$

$r = a(1 - e\cos E)$

$dr = ae\sin E\, dE$

$$= \frac{1}{\cancel{a^{3/2}}} \int \frac{a^{\cancel{2}}\cancel{e}\,\cancel{\sin E}\,(1 - e\cos E)\, dE}{\cancel{a^{1/2}}\,\cancel{e}\,\cancel{\sin E}}$$

$(ae^2 \sin^2 E)^{1/2}$

$$\theta_c = \int (1 - e\cos E)\, dE$$

$$= \int dE - e\int \cos E\, dE$$

$$= E - e\sin E = M$$

$$\theta_c = M$$

DELAUNAY VARIABLES

ANGLE θ	ACTION J
Ω	$[\mu a(1-e^2)]^{1/2}\cos I$
ω	$[\mu a(1-e^2)]^{1/2}$
M	$(\mu a)^{1/2}$

$H = -\frac{\mu}{2a}$

$\dot{\theta}_i = \frac{\partial H}{\partial J_i}$

$\dot{J}_i = -\frac{\partial H}{\partial \theta_i}$