HAMILTONIAN FORMULATION

$$\ddot{r} + \frac{\mu}{r^3} r = 0$$

$$r = (x, y, z)$$

$$\dot{r} = (\dot{x}, \dot{y}, \dot{z})$$

$$r = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

$$p = p_x \hat{i} + p_y \hat{j} + p_z \hat{k}$$

$$p = \frac{m_1 m_2}{(m_1 + m_2)} v \qquad v = \dot{r}$$

$$\dot{r} = \nabla_p H \qquad \dot{p} = -\nabla_r H$$

$$\bar{\nabla}_p \equiv \hat{i} \frac{\partial}{\partial p_x} + \hat{j} \frac{\partial}{\partial p_y} + \hat{k} \frac{\partial}{\partial p_z}$$

$$\bar{\nabla}_r \equiv \hat{i} \frac{\partial}{\partial r_x} + \hat{j} \frac{\partial}{\partial r_y} + \hat{k} \frac{\partial}{\partial r_z}$$

$$H = \frac{p^2}{2\mu^*} - \frac{\mu \mu^*}{r}$$

$$\mu = G(m_1 + m_2)$$

$$\mu^* = \frac{m_1 m_2}{(m_1 + m_2)}$$ REDUCED MASS

$$\dot{r} = \nabla_p H$$
$$\dot{p} = -\nabla_r H$$

$$H = \frac{p^2}{2\mu^*} - \frac{\mu\mu^*}{r}$$

$$\dot{r} = \frac{p}{\mu^*} \qquad \dot{p} = -\frac{\mu\mu^*}{r^3} r$$

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C$$

$$\boxed{H = \mu^* C}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

HAMILTON'S EQUATIONS

$$H = H(q_i, p_i, t)$$

ANGLE-ACTION VARIABLES

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

(q_i, p_i)

DELAUNAY VARIABLES

$l = M \qquad L = (\mu a)^{1/2} = E$ (ENERGY) .

$g = \omega \qquad G = L(1-e^2)^{1/2} = h$ (ANGULAR MOMENTUM)

$h = \Omega \qquad H = G \cos I = h_z$ (VERTICAL COMPONENT OF ANGULAR MOMENTUM)

$\theta = [l, g, h]$

$J = [L, G, H]$

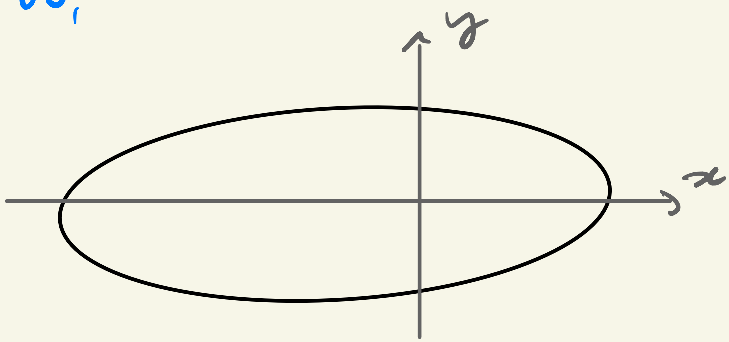
$$\dot{J}_i = -\frac{\partial H}{\partial \theta_i} \quad ; \quad \dot{\theta}_i = \frac{\partial H}{\partial J_i}$$

ANGLE-ACTION VARIABLES

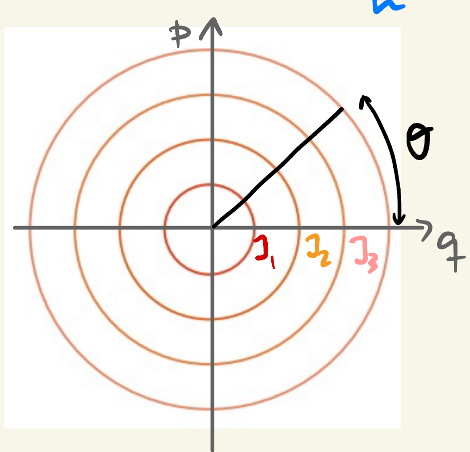
(θ, J) $\qquad \dot{J}_i = -\dfrac{\partial H}{\partial \theta_i}$; $\dot{\theta}_i = \dfrac{\partial H}{\partial J_i}$

$\dot{J}_i = -\dfrac{\partial H}{\partial \theta_i} = 0$ → HAMILTONIAN IS INDEPENDENT OF ANGLES θ, $H = H(J)$

$\dot{\theta}_i = \dfrac{\partial H}{\partial J_i} = \Omega(J) \equiv$ const $\qquad \boxed{\theta_i(t) = \theta_i(0) + \Omega_i t}$

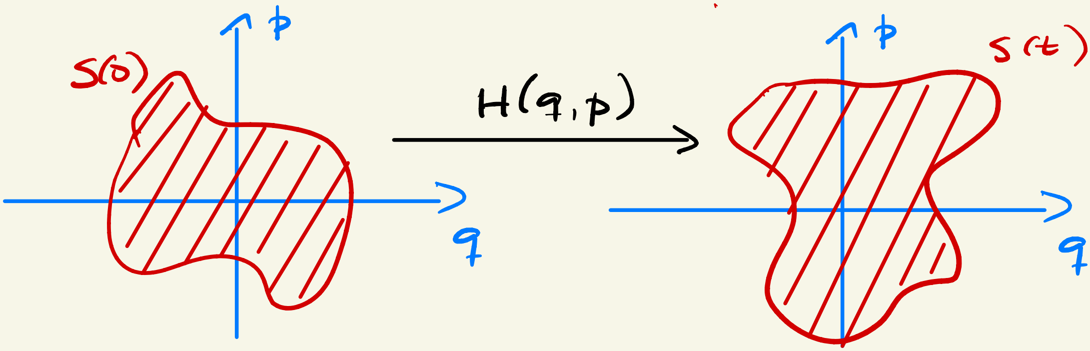


$$X(\theta, J) = \sum_n X_n(J)\, e^{i n \cdot \theta}$$



$(q_i, p_i) \Leftrightarrow (\theta_i, J_i)$

Poincaré Invariant



$$\boxed{\iint_{S(0)} dq \cdot dp = \iint_{S(t)} dq \cdot dp}$$

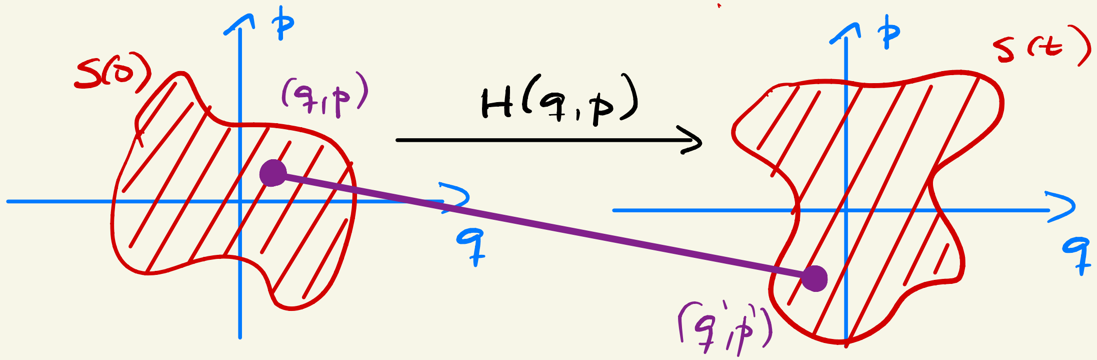
$$A = \iint_{S(t)} dq \cdot dp \qquad \begin{aligned} q &= q(u,v) \\ p &= p(u,v) \end{aligned}$$

$$= \iint_{S(t)} \frac{\partial(q,p)}{\partial(u,v)}\, du\, dv$$

$$\frac{\partial(q,p)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \\ \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \end{vmatrix} \qquad \frac{dA}{dt} = ?$$

$$\frac{dA}{dt} = ?$$

$$t' \equiv t + \delta t \qquad (q', p') = H(q, p)$$



$$q' = q + \frac{\partial q}{\partial t}\delta t = q + \frac{\partial H}{\partial p}\delta t$$

$$p' = p + \frac{\partial}{\partial t}\delta t = p - \frac{\partial H}{\partial q}\delta t$$

$$A(t) = \iint_{S(t)} \frac{\partial(q, p)}{\partial(u, v)}\, du\, dv$$

$$A'(t) = \iint_{S(t')} \frac{\partial(q', p')}{\partial(u, v)}\, du\, dv$$

$$\frac{dA}{dt} = \lim_{\delta t \to 0} \frac{1}{\cancel{\delta t}} \times$$

$$\iint \left(\frac{\partial q}{\partial v} \frac{\partial^2 H}{\partial u \partial q} - \frac{\partial p}{\partial u} \frac{\partial^2 H}{\partial v \partial p} + \frac{\partial p}{\partial v} \frac{\partial^2 H}{\partial u \partial p} - \frac{\partial q}{\partial u} \frac{\partial^2 H}{\partial v \partial q} \right) \cancel{\delta t} \, du \, dv$$

$$\frac{dA}{dt} = \iint \left(\frac{\partial q}{\partial v} \frac{\partial^2 H}{\partial u \partial q} - \frac{\partial p}{\partial u} \frac{\partial^2 H}{\partial v \partial p} + \frac{\partial p}{\partial v} \frac{\partial^2 H}{\partial u \partial p} - \frac{\partial q}{\partial u} \frac{\partial^2 H}{\partial v \partial q} \right) du \, dv$$

$$\frac{\partial}{\partial u} = \frac{\partial q}{\partial u} \frac{\partial}{\partial q} + \frac{\partial p}{\partial u} \frac{\partial}{\partial p}$$

$$\frac{\partial}{\partial v} = \frac{\partial q}{\partial v} \frac{\partial}{\partial q} + \frac{\partial p}{\partial v} \frac{\partial}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\frac{\partial H}{\partial u} = \frac{\partial q}{\partial u} \frac{\partial H}{\partial q} + \frac{\partial p}{\partial u} \frac{\partial H}{\partial p}$$

$$\frac{\partial H}{\partial u} = -\frac{\partial q}{\partial u} \dot{p} + \frac{\partial p}{\partial u} \dot{q}$$

$$\frac{\partial H}{\partial v} = -\frac{\partial q}{\partial v} \dot{p} + \frac{\partial p}{\partial v} \dot{q}$$

$$\frac{\partial q}{\partial v}\cancel{\frac{\partial^2 H}{\partial u \partial q}}^{0} - \frac{\partial p}{\partial u}\cancel{\frac{\partial^2 H}{\partial v \partial p}} + \frac{\partial p}{\partial v}\cancel{\frac{\partial^2 H}{\partial u \partial p}} - \frac{\partial q}{\partial u}\cancel{\frac{\partial^2 H}{\partial v \partial q}} = 0$$

$$\begin{cases} \dfrac{\partial H}{\partial u} = -\dfrac{\partial q}{\partial u}\dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial u}\dfrac{\partial q}{\partial t} \\[2ex] \dfrac{\partial H}{\partial v} = -\dfrac{\partial q}{\partial v}\dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial v}\dfrac{\partial q}{\partial t} \end{cases}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial H}{\partial u}\right) = \frac{\partial}{\partial q}\left(-\frac{\partial q}{\partial u}\frac{\partial p}{\partial t} + \frac{\partial p}{\partial u}\frac{\partial q}{\partial t}\right)$$

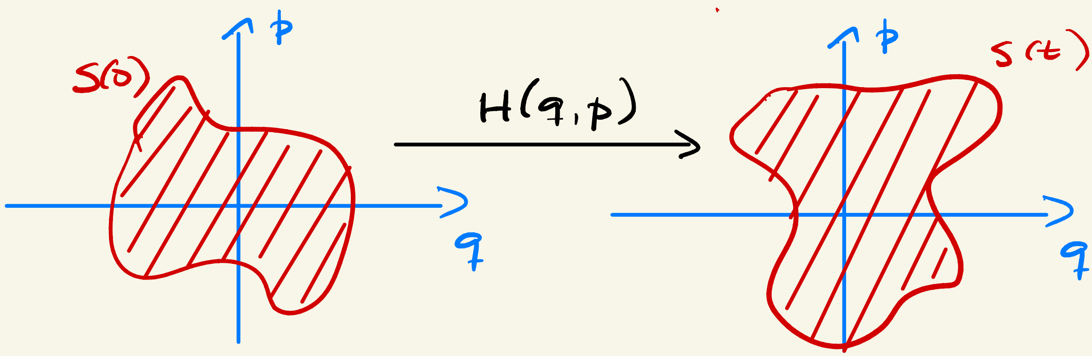
$$= -\frac{\partial p}{\partial t}\left(\frac{\partial}{\partial q}\frac{\partial q}{\partial u}\right) - \frac{\partial q}{\partial u}\left(\frac{\partial}{\partial q}\frac{\partial p}{\partial t}\right) + \frac{\partial p}{\partial u}\left(\frac{\partial}{\partial q}\frac{\partial q}{\partial t}\right) + \frac{\partial q}{\partial t}\left(\frac{\partial}{\partial q}\frac{\partial p}{\partial u}\right)$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

$$= \cancel{-\frac{\partial p}{\partial t}\left(\frac{\partial}{\partial u}\overbrace{\frac{\partial q}{\partial q}}^{=1}\right)} - \cancel{\frac{\partial q}{\partial u}\left(\frac{\partial}{\partial t}\frac{\partial p}{\partial q}\right)} + \cancel{\frac{\partial p}{\partial u}\left(\frac{\partial}{\partial t}\overbrace{\frac{\partial q}{\partial q}}^{=1}\right)} + \cancel{\frac{\partial q}{\partial t}\left(\frac{\partial}{\partial u}\frac{\partial p}{\partial q}\right)}$$

$$\frac{dA}{dt} = \iint \left(\frac{\partial q}{\partial v}\frac{\partial^2 H}{\partial u \partial q} - \frac{\partial p}{\partial u}\frac{\partial^2 H}{\partial v \partial p} + \frac{\partial p}{\partial v}\frac{\partial^2 H}{\partial u \partial p} - \frac{\partial q}{\partial u}\frac{\partial^2 H}{\partial v \partial q}\right) du\, dv$$

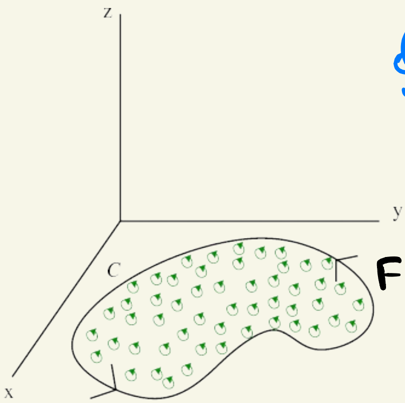
$$= 0$$



$$\boxed{\iint_{S(0)} dq \cdot dp = \iint_{S(t)} dq \cdot dp}$$

GREEN'S THEOREM

STOKES THEOREM



$$\oint_C F \cdot ds = \iint (\nabla \times F)\, dA$$

$$F = f(x,y)\,\hat{x} + g(x,y)\,\hat{y}$$

$$\iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy = \oint_C [f\,dx + g\,dy]$$

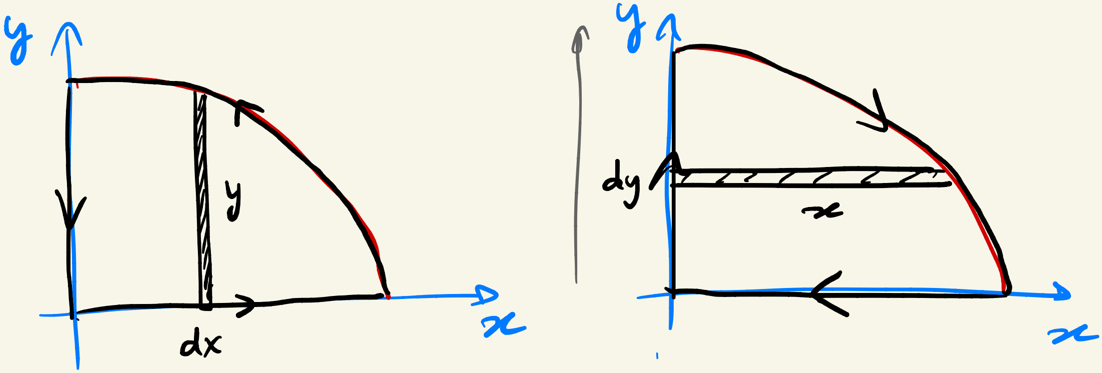
$$f = -y \quad ; \quad g = x$$

$$\oint_C (x\,dy - y\,dx) = 2 \iint dx\,dy$$

$$\oint_C x\,dy - \oint y\,dx = 2 \iint dx\,dy$$

$$\oint_C x\,dy - \oint y\,dx = 2\iint dx\,dy$$

$$\oint_C y\,dx = -\oint_C x\,dy$$



$$\oint_C y\,dx = -\oint_C x\,dy$$

$$\boxed{\oint_C x\,dy = \iint dx\,dy}$$

$$dq\,dp = d\theta\, dJ$$

$$J' = \frac{1}{2\pi}\iint d\theta\, dJ$$

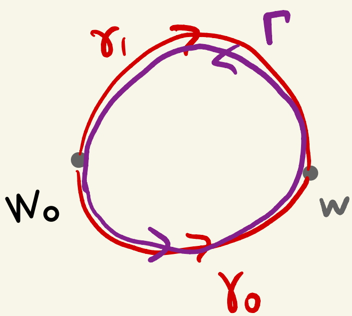
$$\oint_c x\,dy = \iint dx\,dy$$

$$= \frac{1}{2\pi}\oint_c J\, d\theta$$

$$= J$$

$$\boxed{J = \frac{1}{2\pi}\oint p\, dq}$$

HAMILTON-JACOBI EQUATION



q, p

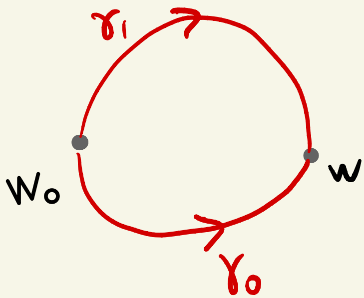
q', p'

$$\oint_\Gamma p\, dq = \oint_\Gamma p'\, dq'$$

$$\oint_\Gamma (p\, dq - p'\, dq') = 0$$

$$\int_{\gamma_0} p\, dq - p'\, dq' - \int_{\gamma_1} p\, dq - p'\, dq' = 0$$

$$\int_{\gamma_0} p\,dq - p'\,dq' = \int_{\gamma_1} p\,dq - p'\,dq'$$



$$S(w) = \int_{w_0}^{w} p\,dq - p'\,dq'$$

$$dS = p\,dq - p'\,dq'$$

$S \equiv$ GENERATING FUNCTION

$$p = \frac{\partial S}{\partial q}$$

$$H(q, p) = E$$

HAMILTON-JACOBI EQUATION

$$H\left(q, \frac{\partial S}{\partial q}\right) = E$$

$$J = \frac{1}{2\pi} \oint_{\gamma} p \, dq \qquad p = \frac{\partial S}{\partial q}$$

$$J = \frac{1}{2\pi} \oint_{\gamma} \frac{\partial S}{\partial q} \, dq$$

$$J = \frac{\Delta S}{2\pi}$$

HARMONIC OSCILLATOR

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} = E \qquad p = \frac{\partial S}{\partial x}$$

$$= \frac{1}{2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \omega^2 x^2\right] = E$$

$$\frac{\partial S}{\partial x} = \left(2E - \omega^2 x^2\right)^{1/2} \qquad 2E = k^2$$

$$\frac{\partial S}{\partial x} = \left(k^2 - \omega^2 x^2\right)^{1/2} \quad \therefore \quad S = \int \left(k^2 - \omega^2 x^2\right)^{1/2} dx$$

$$S = k \int \left(1 - \frac{\omega^2}{k^2} x^2\right)^{1/2} dx \qquad 1 - \cos^2\psi = \sin^2\psi$$

$$\frac{\omega^2}{k^2} x^2 = \cos^2\psi \quad \therefore \quad x = \pm \frac{k}{\omega}\cos\psi$$

$$dx = \mp \frac{k}{\omega}\sin\psi \, d\psi$$

$$S = \frac{k^2}{\omega}\int \sin^2\psi \, d\psi$$

$$\boxed{S = \frac{k^2}{2\omega}\left(\psi - \frac{1}{2}\sin 2\psi\right)}$$

$$S = \frac{k^2}{2\omega}\left(\psi - \frac{1}{2}\sin 2\psi\right)$$

$$J = \frac{\Delta S}{2\pi} = \frac{1}{2\pi}\left[S\Big|_{\psi=2\pi} - S\Big|_{\psi=0} \right]$$

$$= \frac{1}{\cancel{2\pi}}\left[\frac{k^2}{\cancel{2}\omega} \cancel{2\pi} \right]$$

$$J = \frac{k^2}{2\omega} = \frac{(p^2 + \omega^2 x^2)}{2\omega} = \frac{E}{\omega}$$

$$2E = k^2$$

$$H = J \cdot \omega$$

$$\cancel{\dot{J}} = -\frac{\partial H}{\partial \theta} = 0 \quad ; \quad \dot{\theta} = \frac{\partial H}{\partial J} = \omega$$

$$\theta = \frac{\partial S}{\partial J} \qquad S = J\left(\psi - \frac{1}{2}\sin 2\psi\right)$$

$$= \psi - \frac{1}{2}\sin 2\psi + J \frac{\partial}{\partial J}\left(\psi - \frac{1}{2}\sin 2\psi\right)$$

$$= \psi - \frac{1}{2}\sin 2\psi + J\left(1 - \cos 2\psi\right)\frac{\partial \psi}{\partial J}$$

$$\theta = \psi - \frac{1}{2}\sin 2\psi + J(1-\cos 2\psi)\frac{\partial\psi}{\partial J}$$

$$x = -\frac{K}{\omega}\cos\psi \qquad \cos\psi = -\frac{\omega}{k}x \qquad 2E = k^2$$

$$E = J\cdot\omega$$

$$\cos\psi = -\sqrt{\frac{\omega}{2J}}\cdot x$$

$$-\sin\psi\frac{\partial\psi}{\partial J} = -\sqrt{\frac{\omega}{2}}\left(-\frac{1}{2J^{3/2}}\right)\cdot x$$

$$\cancel{-}\sin\psi\frac{\partial\psi}{\partial J} = \sqrt{\frac{\cancel{\omega}}{\cancel{2}}}\left(\cancel{-}\frac{1}{2J^{\cancel{3/2}}}\right)\sqrt{\frac{\cancel{2}\cancel{J}}{\cancel{\omega}}}\cos\psi$$

$$\frac{\partial\psi}{\partial J} = \frac{1}{2J}\frac{\cos\psi}{\sin\psi}$$

$$\sin^2\psi + \cancel{\cos^2\psi} \quad \cancel{-\cos^2\psi} + \sin^2\psi$$

$$\theta = \psi - \frac{1}{2}\sin 2\psi + \cancel{J}(1-\cos 2\psi)\frac{1}{2\cancel{J}}\frac{\cos\psi}{\sin\psi}$$

$$= \psi - \frac{1}{2}\sin 2\psi + \sin\psi\cos\psi$$

$$= \psi - \cancel{\sin\psi\cos\psi} + \cancel{\sin\psi\cos\psi}$$

$$\boxed{\theta = \psi} \qquad \boxed{J = E/\omega}$$