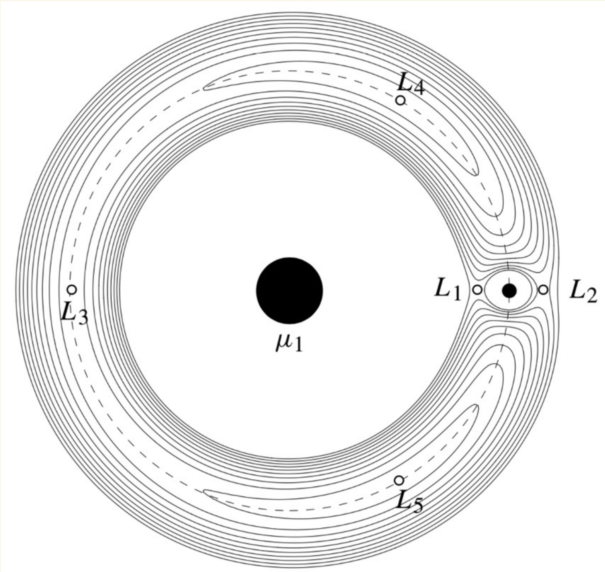
STABILITY OF LAGRANGIAN POINTS



$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y}$$

$n = 1$

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

EQUILIBRIUM POINT (x_0, y_0)

$x = x_0 + X \qquad y = y_0 + Y$

X, Y: SMALL DISPLACEMENT

$$\ddot{X} - 2\dot{Y} = \frac{\partial U}{\partial x} \approx \cancel{\left.\frac{\partial U}{\partial x}\right|_0} + X \left.\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)\right|_0 + Y \left.\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)\right|_0$$

$$\ddot{Y} + 2\dot{X} = \frac{\partial U}{\partial y} \approx \cancel{\left.\frac{\partial U}{\partial y}\right|_0} + Y \left.\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)\right|_0 + X \left.\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)\right|_0$$

$$\left.\frac{\partial U}{\partial x}\right|_0 = 0 \qquad \left.\frac{\partial U}{\partial y}\right|_0 = 0$$

$$\ddot{x} - 2\dot{y} = x \left.\frac{\partial^2 U}{\partial x^2}\right|_0 + y \left.\frac{\partial^2 U}{\partial y \partial x}\right|_0 + O(x^2, y^2, xy)$$

$$\ddot{y} + 2\dot{x} = y \left.\frac{\partial^2 U}{\partial y^2}\right|_0 + x \left.\frac{\partial^2 U}{\partial y \partial x}\right|_0 + O(x^2, y^2, xy)$$

$$U_{xx} = \left.\frac{\partial^2 U}{\partial x^2}\right|_0 \qquad U_{yy} = \left.\frac{\partial^2 U}{\partial y^2}\right|_0 \qquad U_{xy} = \left.\frac{\partial^2 U}{\partial x \partial y}\right|_0$$

$$\ddot{x} - 2\dot{y} = x U_{xx} + y U_{xy}$$

$$\ddot{y} + 2\dot{x} = x U_{xy} + y U_{yy}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{bmatrix} \qquad |A - \lambda I| = 0$$

$$|A - \lambda I| = 0$$

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0$$

$$\lambda_{1,2} = \pm\left\{\frac{1}{2}(U_{xx} + U_{yy} - 4) - \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)\right]^{1/2}\right\}^{1/2}$$

$$\lambda_{3,4} = \pm\left\{\frac{1}{2}(U_{xx} + U_{yy} - 4) + \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)\right]^{1/2}\right\}^{1/2}$$

$$\lambda_{1,2} = \pm(j_1 + i k_1) \qquad \lambda_{3,4} = \pm(j_2 + i k_2)$$

$$X = \sum_{j=1}^{4} \alpha_j e^{\lambda_j t} \qquad Y = \sum_{j=1}^{4} \beta_j e^{\lambda_j \cdot t}$$

IF $j \neq 0$ GROWTH UNSTABLE

IF $j = 0$ OSCILLATION STABLE

COLLINEAR POINTS

$$U = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$\frac{\partial U}{\partial x} = x + \mu_1 \frac{\partial}{\partial x}\left(1/r_1\right) + \mu_2 \frac{\partial}{\partial x}\left(1/r_2\right)$$

$$\frac{\partial U}{\partial y} = y + \mu_1 \frac{\partial}{\partial y}\left(1/r_1\right) + \mu_2 \frac{\partial}{\partial y}\left(1/r_2\right)$$

$$\frac{\partial^2 U}{\partial x^2} = 1 + \mu_1 \frac{\partial^2}{\partial x^2}\left(\frac{1}{r_1}\right) + \mu_2 \frac{\partial^2}{\partial x^2}\left(\frac{1}{r_2}\right)$$

$$\frac{\partial^2 U}{\partial y^2} = 1 + \mu_1 \frac{\partial^2}{\partial y^2}\left(\frac{1}{r_1}\right) + \mu_2 \frac{\partial^2}{\partial y^2}\left(\frac{1}{r_2}\right)$$

$$\frac{\partial^2 U}{\partial x \partial y} = \mu_1 \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{r_1}\right) + \mu_2 \frac{\partial^2}{\partial x \partial y}\left(1/r_2\right)$$

$$r_1 = (x+\mu_2)^2 + y^2$$
$$r_2 = (x-\mu_1)^2 + y^2$$

$$\therefore \frac{\partial^2 U}{\partial x \partial y} = 0$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r_1}\right) = -\frac{1}{r_1^2}\frac{\partial r_1}{\partial y} = -\frac{1}{r_1^2} 2y \qquad \therefore \frac{\partial^2}{\partial y^2} = +\frac{4y}{r_1^3} - \frac{2}{r_1^2}$$

$$X = \sum_{j=1}^{4} \alpha_j e^{\lambda_j t} \qquad Y = \sum_{j-1}^{4} \beta_j e^{\lambda_j \cdot t}$$

$$r_1 = (x+\mu_2)^2 + y^2$$

$$r_2 = (x-\mu_1)^2 + y^2$$

$$U = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$\frac{\partial U}{\partial x} = x + \mu_1 \frac{\partial}{\partial x}\left(1/r_1\right) + \mu_2 \frac{\partial}{\partial x}\left(1/r_2\right)$$

$$\frac{\partial U}{\partial y} = y + \mu_1 \frac{\partial}{\partial y}\left(1/r_1\right) + \mu_2 \frac{\partial}{\partial y}\left(1/r_2\right)$$

STABILITY OF LAGRANGIAN POINTS

$$X = \sum_{j=1}^{4} \alpha_j e^{\lambda_j t} \qquad Y = \sum_{j=1}^{4} \beta_j e^{\lambda_j \cdot t}$$

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx} U_{yy} - U_{xy}^2 = 0$$

$$U_{xx} = \left.\frac{\partial^2 U}{\partial x^2}\right|_0 \qquad U_{yy} = \left.\frac{\partial^2 U}{\partial y^2}\right|_0 \qquad U_{xy} = \left.\frac{\partial^2 U}{\partial x \partial y}\right|_0$$

$$\lambda_{1,2} = \pm \left\{ \frac{1}{2}(U_{xx} + U_{yy} - 4) - \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx} U_{yy} - U_{xy}^2)\right]^{1/2} \right\}^{1/2}$$

$$\lambda_{3,4} = \pm \left\{ \frac{1}{2}(U_{xx} + U_{yy} - 4) + \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx} U_{yy} - U_{xy}^2)\right]^{1/2} \right\}^{1/2}$$

$$\lambda_{1,2} = \pm (j_1 + i k_1) \qquad \lambda_{3,4} = \pm (j_2 + i k_2)$$

U_{xx}

$$U = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$r_1^2 = (x + \mu_2)^2 + y^2$$

$$r_2^2 = (x - \mu_1)^2 + y^2$$

$$U = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$\frac{\partial U}{\partial x} = x + \mu_1 \frac{\partial}{\partial x}\left(\frac{1}{r_1}\right) + \mu_2 \frac{\partial}{\partial x}\left(\frac{1}{r_2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right) = -\frac{1}{r_1^2}\frac{\partial r_1}{\partial x}$$

$$r_1^2 = (x+\mu_2)^2 + y^2$$

$$\frac{\partial r_1}{\partial x} = \frac{(x+\mu_2)}{r_1} \; ; \; \frac{\partial r_1}{\partial y} = \frac{y}{r_1}$$

$$\frac{\partial r_2}{\partial x} = \frac{(x-\mu_1)}{r_2} \; ; \; \frac{\partial r_2}{\partial y} = \frac{y}{r_2}$$

$$\frac{\partial U}{\partial x} = x - \frac{\mu_1(x+\mu_2)}{r_1^3} - \frac{\mu_2(x-\mu_1)}{r_2^3}$$

$$\boxed{\frac{\partial^2 U}{\partial x^2} = 1 + \frac{3\mu_1(x+\mu_2)}{r_1^4} + \frac{3\mu_2(x-\mu_1)}{r_2^4} - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}}$$

U_{yy}

$$\boxed{\frac{\partial^2 U}{\partial y^2} = 1 + \frac{3\mu_1 y}{r_1^4} + \frac{3\mu_2 y}{r_2^4} - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}}$$

U_{xy}

$$\frac{\partial U}{\partial y} = y - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3}$$

$$\frac{\partial U}{\partial y} = y - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3}$$

$$\boxed{\frac{\partial^2 U}{\partial x \partial y} = 3y\left[\frac{\mu_1 (x+\mu_2)}{r_1^5} + \frac{\mu_2 (x-\mu_1)}{r_2^5}\right]}$$

SOLUTIONS:

$$\frac{\partial^2 U}{\partial x^2} = 1 + \frac{3\mu_1 (x+\mu_2)}{r_1^4} + \frac{3\mu_2 (x-\mu_1)}{r_2^4} - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}$$

$$\frac{\partial^2 U}{\partial y^2} = 1 + \frac{3\mu_1 y}{r_1^4} + \frac{3\mu_2 y}{r_2^4} - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}$$

$$\frac{\partial^2 U}{\partial x \partial y} = 3y\left[\frac{\mu_1 (x+\mu_2)}{r_1^5} + \frac{\mu_2 (x-\mu_1)}{r_2^5}\right]$$

COLLINEAR POINTS L_1, L_2, L_3 $\quad y = 0$

$$\frac{\partial^2 U}{\partial x^2} = 1 + \frac{3\mu_1 (\cancel{x+\mu_2})}{r_1^{\cancel{4}\,3}} + \frac{3\mu_2 (\cancel{x-\mu_1})}{r_2^{\cancel{4}\,3}} - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3} = 1 + \frac{2\mu_1}{r_1^3} + \frac{2\mu_2}{r_2^3}$$

$$\frac{\partial^2 U}{\partial y^2} = 1 + \cancel{\frac{3\mu_1 y}{r_1^4}} + \cancel{\frac{3\mu_2 y}{r_2^4}} - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3} = 1 - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}$$

$$\frac{\partial^2 U}{\partial x \partial y} = 3y\,\cancel{\left[\frac{\mu_1 (x+\mu_2)}{r_1^5} + \frac{\mu_2 (x-\mu_1)}{r_2^5}\right]} = 0$$

$U_{xx} = 1 + 2A \qquad U_{yy} = 1 - A \qquad U_{xy} = 0$

$$A = \frac{\mu_1}{(r_1)_0^3} + \frac{\mu_2}{(r_2)_0^3}$$

$$\lambda = \pm\left\{\frac{1}{2}\left(U_{xx} + U_{yy} - 4\right) \pm \frac{1}{2}\left[\left(4 - U_{xx} - U_{yy}\right)^2 - 4\left(U_{xx}U_{yy}\right)\right]^{1/2}\right\}^{1/2}$$

$A = 2$

$$\lambda = \pm\left\{\underline{\underline{\frac{1}{2}(A-2)}} \pm \frac{1}{2}\left[9A^2 - 8A\right]^{1/2}\right\}^{1/2}$$

$L_1, L_2, L_3 \qquad \lambda = \pm(j + ik) \qquad j \neq 0$ $\boxed{\text{UNSTABLE}}$

TRIANGULAR POINTS L_4, L_5

$$\frac{\partial^2 U}{\partial x^2} = 1 + \frac{3\mu_1(x+\mu_2)}{\cancel{r_1^4}} + \frac{3\mu_2(x-\mu_1)}{\cancel{r_2^4}} - \frac{\mu_1}{\cancel{r_1^3}} - \frac{\mu_2}{\cancel{r_2^3}} = \frac{3}{4}$$

$$\frac{\partial^2 U}{\partial y^2} = 1 + \frac{3\mu_1 y}{\cancel{r_1^4}} + \frac{3\mu_2 y}{\cancel{r_2^4}} - \frac{\mu_1}{\cancel{r_1^3}} - \frac{\mu_2}{\cancel{r_2^3}} = \frac{9}{4}$$

$$\frac{\partial^2 U}{\partial x \partial y} = 3y\left[\frac{\mu_1(x+\mu_2)}{\cancel{r_1^5}} + \frac{\mu_2(x-\mu_1)}{\cancel{r_2^5}}\right] = \pm\frac{3\sqrt{3}}{4}(1-2\mu_2)$$

$r_1 = r_2 = 1$

$x = \frac{1}{2} - \mu_2$

$y = \pm\frac{\sqrt{3}}{2}$

$$U_{xx} = \frac{3}{4} \qquad U_{yy} = \frac{9}{4} \qquad U_{xy} = \pm \frac{3\sqrt{3}}{4}(1-2\mu_2)$$

$$\lambda = \pm\left\{\frac{1}{2}\left(U_{xx}+U_{yy}-4\right) \pm \frac{1}{2}\left[\left(4-U_{xx}-U_{yy}\right)^2 - 4\left(U_{xx}U_{yy}-U_{xy}^2\right)\right]^{1/2}\right\}^{1/2}$$

$$U_{xx}+U_{yy}-4 = \frac{3}{4}+\frac{9}{4}-\frac{16}{4} = -1$$

$$\lambda = \pm\left\{-\frac{1}{2} \pm \frac{1}{2}\left[1 - \cancel{4\left(U_{xx}U_{yy}-U_{xy}^2\right)}\right]^{1/2}\right\}^{1/2}$$

$$U_{xx}U_{yy} - U_{xy}^2 = \frac{27}{16} - \frac{27}{16}\left(1-\cancel{2\mu_2}\right)^2 = 0 \qquad \mu_2 \ll 1$$

$$\lambda = \pm\left(-\frac{1}{2} \pm \frac{1}{2}\right)^{1/2} = 0\ ;\ \pm i \qquad j=0$$

L_4, L_5 $\boxed{\text{STABLE}}$

MOTION AROUND L4 AND L5

$$\lambda = \pm \frac{1}{\sqrt{2}} \sqrt{-1 \pm \sqrt{1 - 27(1-\mu_2)\mu_2}}$$

$$1 - 27(1-\mu_2)\mu_2 \geq 0 \quad \text{(STABILITY)}$$

$$\mu_2 \leq \frac{27 - \sqrt{621}}{54} \approx 0.04$$

FOR $\mu_2 << 1 \neq 0$

$$\sqrt{1 - 27(1-\mu_2)\mu_2} \approx \sqrt{1 - 27\mu_2} \approx 1 - \frac{27}{2}\mu_2$$

$$\lambda = \pm \frac{1}{\sqrt{2}} \sqrt{-1 \pm \left(1 - \frac{27}{2}\mu_2\right)} = \pm \sqrt{-\frac{1}{2} \pm \left(\frac{1}{2} - \frac{27}{4}\mu_2\right)}$$

$$\boxed{\lambda_{1,2} = \pm \sqrt{-1 - \frac{27}{4}\mu_2} \approx \pm i}$$

$$\boxed{\lambda_{3,4} = \pm \sqrt{-\frac{27\mu_2}{4}}}$$

LIBRATION

$$X = \sum_{j=1}^{4} \alpha_j e^{\lambda_j t} \qquad Y = \sum_{j=1}^{4} \beta_j e^{\lambda_j \cdot t}$$