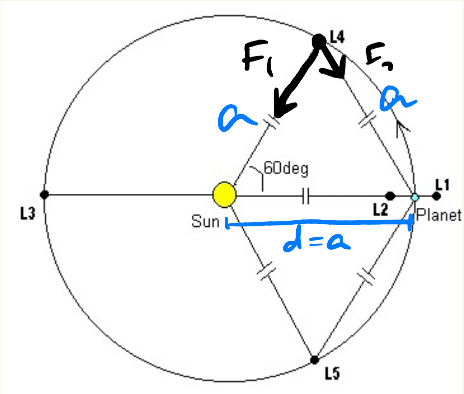
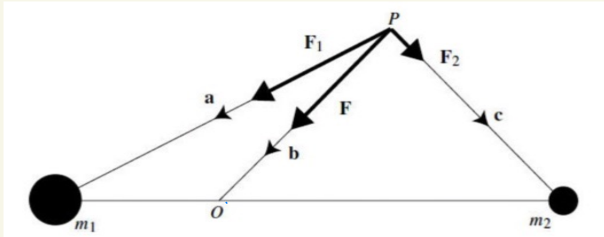
# LAGRANGIAN EQUILIBRIUM POINTS




$$F = F_1 + F_2$$

$$b = \frac{m_1 a + m_2 c}{m_1 + m_2} \qquad m_1 b + m_2 b = m_1 a + m_2 c$$

$$m_1 (a - b) = m_2 (b - c) \qquad \times F \qquad b \times F = 0$$

$$m_1 \, a \times F = -m_2 \, c \times F \qquad F = F_1 + F_2$$

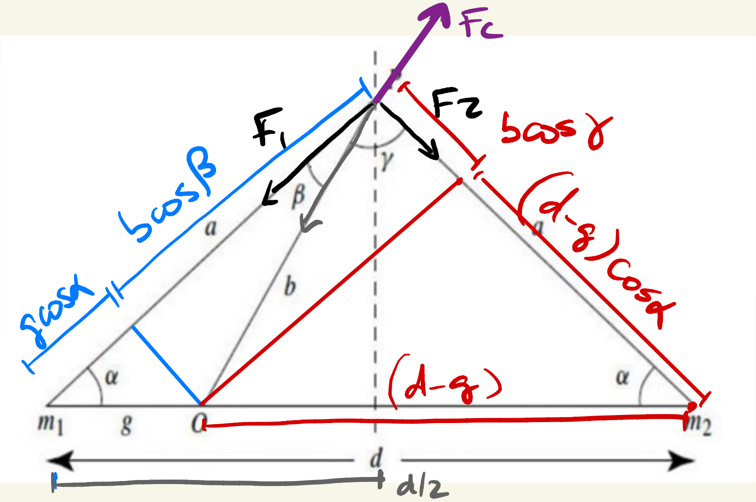
$$m_1 (\cancel{a \times F_1} + a \times F_2) = -m_2 (c \times F_1 + \cancel{c \times F_2})$$

$$m_1 \, a \times F_2 = -m_2 \, c \times F_1$$

$$m_1 a F_2 = m_2 c F_1 \qquad F_1 = \frac{G m_1}{a^2} \qquad F_2 = \frac{G m_2}{c^2}$$

$$\cancel{G m_1 m_2} \frac{a}{c^2} = \cancel{G m_1 m_2} \frac{c}{a^2}$$

$$\boxed{a = c}$$




$$F_c = \Omega \times \Omega \times r$$

$$n^2 b$$

$$n^2 b = F_1 \cos\beta + F_2 \cos\gamma$$

$$F_1 = \frac{Gm_1}{a^2}$$

$$F_2 = \frac{Gm_2}{a^2}$$

$$n^2 b = \frac{G}{a^2}\left(m_1 \cos\beta + m_2 \cos\gamma\right)$$

$$\begin{cases} a = b\cos\beta + g\cos\alpha \\ a = b\cos\gamma + (d-g)\cos\alpha \end{cases}$$

$$\cos\alpha = \frac{d}{2a}$$

$$g = \frac{m_2}{(m_1+m_2)} \cdot d \qquad d-g = \frac{m_1}{(m_1+m_2)} d$$

$$\cos\beta = \frac{a - g\cos\alpha}{b} = \frac{a}{b} - \frac{g \cdot d}{2ab}$$

$$b\cos\beta = a - \frac{gd}{2a} = a - \frac{m_2 d^2}{2a(m_1+m_2)}$$

$$b\cos\beta = a - \frac{gd}{2a} = a - \frac{m_2 d^2}{2a(m_1+m_2)}$$

$$b\cos\gamma = a - (d-g)\cos\alpha$$

$$= a - \frac{m_1 d^2}{2a(m_1+m_2)}.$$

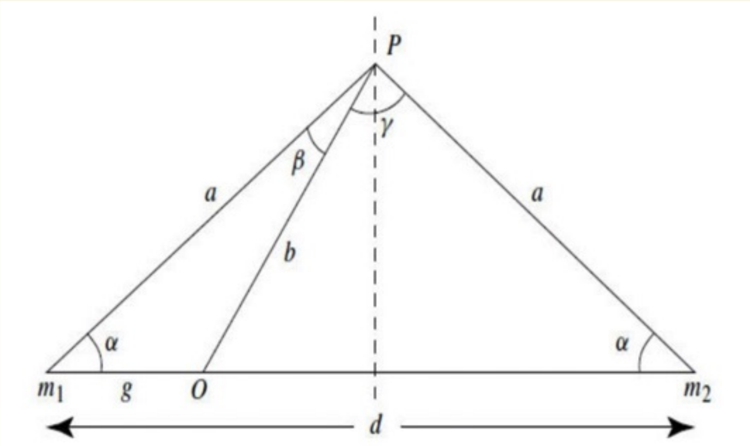
$$n^2 b = \frac{G}{a^2}\left(m_1\cos\beta + m_2\cos\gamma\right)$$

$$n^2 = \frac{G}{a^2b^2}\left(m_1 b\cos\beta + m_2 b\cos\gamma\right)$$

$$= \frac{G}{a^2b^2}\left[m_1 a - \frac{m_1 m_2 d^2}{2a(m_1+m_2)} + m_2 a - \frac{m_1 m_2 d^2}{2a(m_1+m_2)}\right]$$

$$= \frac{G}{a^2b^2}\left[(m_1+m_2)a - \frac{m_1 m_2 d^2}{a(m_1+m_2)}\right]$$

$$= \frac{G(m_1+m_2)}{a^3b^2}\left[a^2 - \frac{m_1 m_2}{(m_1+m_2)^2}d^2\right]$$




$$b^2 = a^2 + g^2 - 2ag\cos\alpha$$

$$= a^2 + g^2 - g \cdot d$$

$$= a^2 + g(g-d)$$

$$= a^2 - \frac{m_1 m_2}{(m_1+m_2)^2} \cdot d^2$$

$$\cos\alpha = \frac{d}{2a}$$

$$g = \frac{m_2}{(m_1+m_2)} \cdot d$$

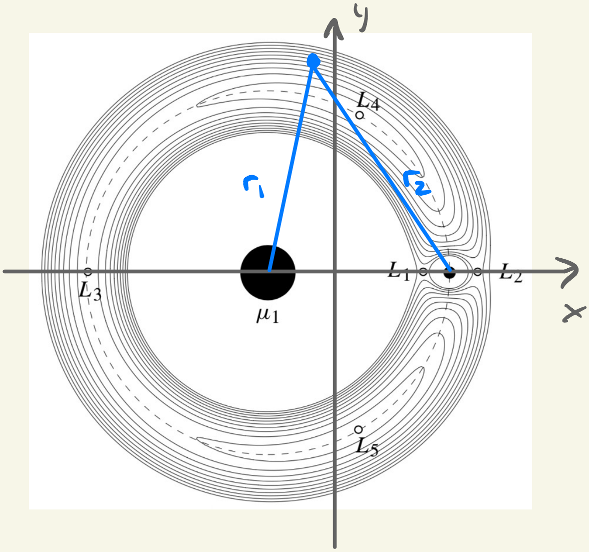
$$d-g = \frac{m_1}{(m_1+m_2)} \cdot d$$

$$n^2 = \frac{G(m_1+m_2)}{a^3 \cancel{b^2}} \cancel{\left[ a^2 - \frac{m_1 m_2}{(m_1+m_2)^2} d^2 \right]}$$

$$n^2 = \frac{G(m_1+m_2)}{a^3} = \frac{G(m_1+m_2)}{d^3}$$

$$\boxed{a=d}$$

# LAGRANGIAN POINTS




$$C_J = 2U - \cancel{v^2}$$

$\cancel{v^2} = 0$ (EQUILIBRIUM)

$$U = \frac{n^2}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$\mu_1 = Gm_1$$
$$\mu_2 = Gm_2$$
$$\mu_1 + \mu_2 = 1$$
$$a = 1 \quad n = 1$$

$$C_J = 2U$$

$$\frac{d}{dt}C_J = 2\frac{d}{dt}U = 0 \qquad \frac{dU}{dt} = 0$$

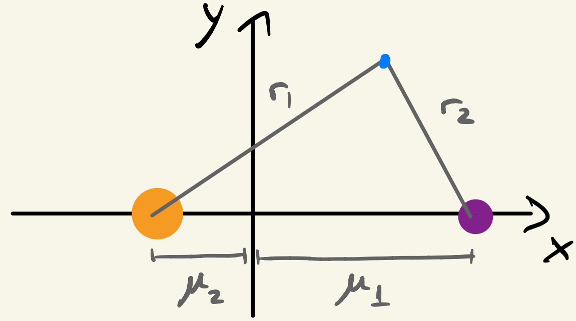
$$\frac{dU}{dt} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}$$

$$\cancel{\ddot{x}} - 2\cancel{n\dot{y}} = \frac{\partial U}{\partial x} = 0$$

$$\cancel{\ddot{y}} + 2\cancel{n\dot{x}} = \frac{\partial U}{\partial y} = 0$$

$$\boxed{\begin{array}{c} U = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \\ \frac{\partial U}{\partial x} = 0 \qquad \frac{\partial U}{\partial y} = 0 \end{array}}$$

$$U = \frac{1}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$




$$\frac{\partial U}{\partial x} = 0 \qquad \frac{\partial U}{\partial y} = 0$$

$$\times\mu_1 \qquad r_1^2 = (x+\mu_2)^2 + y^2$$

$$\times\mu_2 \qquad r_2^2 = (x-\mu_1)^2 + y^2$$

$$\begin{aligned} \mu_1 r_1^2 &= \mu_1(x+\mu_2)^2 + \mu_1 y^2 \\ +\mu_2 r_2^2 &= \mu_2(x-\mu_1)^2 + \mu_2 y^2 \end{aligned}$$

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$$\begin{aligned} \mu_1 r_1^2 + \mu_2 r_2^2 &= \mu_1(x^2 + \cancel{2x\mu_2} + \mu_2^2) + \mu_1 y^2 \\ &\quad + \mu_2(x^2 - \cancel{2x\mu_1} + \mu_1^2) + \mu_2 y^2 \\ &= \cancelto{1}{(\mu_1+\mu_2)}(x^2+y^2) + \underbrace{\mu_1\mu_2^2 + \mu_2\mu_1^2}_{\mu_1\mu_2\cancelto{1}{(\mu_1+\mu_2)}} \end{aligned}$$

$$\mu_1 r_1^2 + \mu_2 r_2^2 = x^2+y^2 + \mu_1\mu_2$$

$$x^2+y^2 = \mu_1 r_1^2 + \mu_2 r_2^2 - \mu_1\mu_2$$

$$U = \frac{1}{2}\left[\mu_1 r_1^2 + \mu_2 r_2^2 - \mu_1\mu_2\right] + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$U = \frac{1}{2}\left[\mu_1 r_1^2 + \mu_2 r_2^2 - \mu_1\mu_2\right] + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

$$\boxed{U = \mu_1\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) + \mu_2\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right) - \frac{\mu_1\mu_2}{2}}$$

$U(r_1, r_2)$

$$\frac{\partial U}{\partial x} = \boxed{\frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x} = 0}$$

$$\frac{\partial U}{\partial y} = \boxed{\frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial y} = 0}$$

$$\boxed{\frac{\partial U}{\partial r_1} = \mu_1\left(-\frac{1}{r_1^2} + r_1\right)} \qquad \boxed{\frac{\partial U}{\partial r_2} = \mu_2\left(-\frac{1}{r_2^2} + r_2\right)}$$

$$r_1^2 = (x + \mu_2)^2 + y^2$$

$$r_2^2 = (x - \mu_1)^2 + y^2$$

$$\boxed{\frac{\partial r_1}{\partial x} = \frac{(x + \mu_2)}{r_1}}$$

$$\boxed{\frac{\partial r_2}{\partial x} = \frac{(x - \mu_1)}{r_2}}$$

$$\boxed{\frac{\partial r_1}{\partial y} = \frac{y}{r_1}} \qquad \boxed{\frac{\partial r_2}{\partial y} = \frac{y}{r_2}}$$

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)\frac{(x+\mu_2)}{r_1} + \mu_2\left(-\frac{1}{r_2^2}+r_2\right)\frac{(x-\mu_1)}{r_2} = 0$$

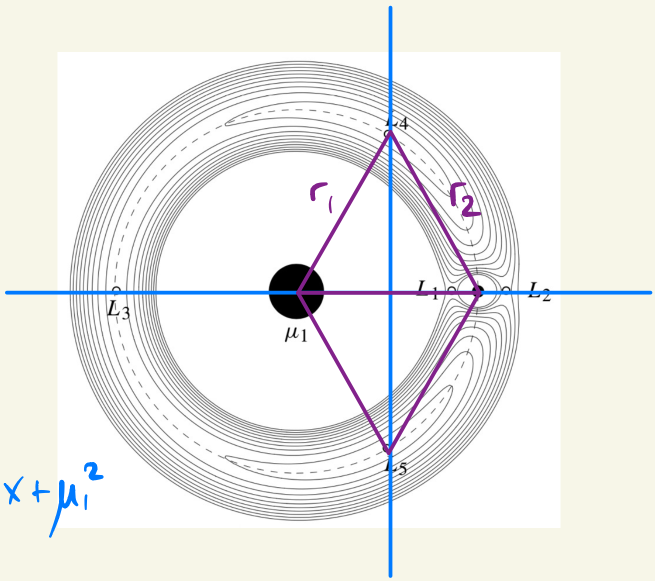
$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)\frac{y}{r_1} + \mu_2\left(-\frac{1}{r_2^2}+r_2\right)\frac{y}{r_2} = 0$$

FIRST SOLUTION:

$$\frac{\partial U}{\partial r_1} = \mu_1\left(-\frac{1}{r_1^2}+r_1\right) = 0 \qquad r_1 = 1$$

$$\frac{\partial U}{\partial r_2} = \mu_2\left(-\frac{1}{r_2^2}+r_2\right) = 0 \qquad r_2 = 1$$

$$r_1 = r_2 = 1$$




$$(x+\mu_2)^2 + y^2 = 1$$
$$(x-\mu_1)^2 + y^2 = 1$$

$$(x+\mu_2)^2 = (x-\mu_1)^2$$

$$\cancel{x^2} + 2\mu_2 x + \mu_2^2 = \cancel{x^2} - 2\mu_1 x + \mu_1^2$$

$$2\overbrace{(\mu_1+\mu_2)}^{1}x = \mu_1^2 - \mu_2^2$$

$$2x = \mu_1^2 - \mu_2^2 + 2\mu_2^2 - 2\mu_2^2$$

$$= \mu_1^2 + \mu_2^2 - 2\mu_2^2 + 2\mu_1\mu_2 - 2\mu_1\mu_2$$

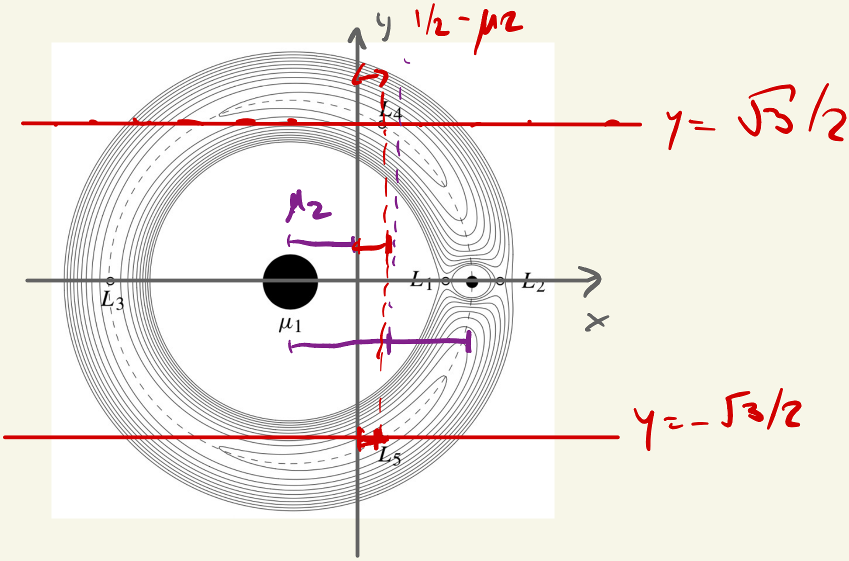
$$= (\mu_1+\mu_2)^2 - 2\mu_2^2 - 2\mu_1\mu_2$$

$$= (\overset{1}{\cancel{\mu_1+\mu_2}})^2 - 2\mu_2(\overset{1}{\cancel{\mu_1+\mu_2}})$$

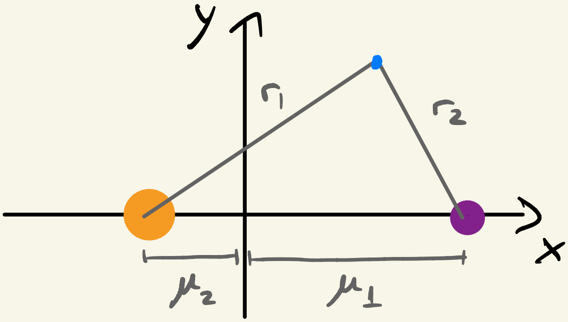
$$2x = 1 - 2\mu_2 \qquad \boxed{x = \frac{1}{2} - \mu_2}$$

$$(x+\mu_2)^2 + y^2 = 1$$

$$\frac{1}{2^2} + y^2 = 1 \quad \therefore \quad \boxed{y = \pm\frac{\sqrt{3}}{2}} \qquad 60^\circ$$




$$x = \frac{1}{2} - \mu_2$$




$$\mu_1 = G m_1$$

$$\mu_2 = G m_2$$

$$r = \frac{r_1 m_1 + r_2 m_2}{m_1 + m_2}$$

$$\boxed{r_1 = \frac{-\mu_2}{(\mu_1+\mu_2)} \cdot r} \qquad \boxed{r_2 = \frac{+\mu_1}{(\mu_1+\mu_2)} \cdot r}$$

UNITS $\underline{r = 1}$

$$\underline{(\mu_1+\mu_2) = 1}$$

$$r_1 = -\mu_2$$

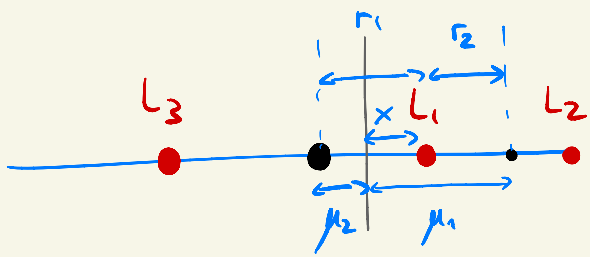
$$r_2 = \mu_1$$

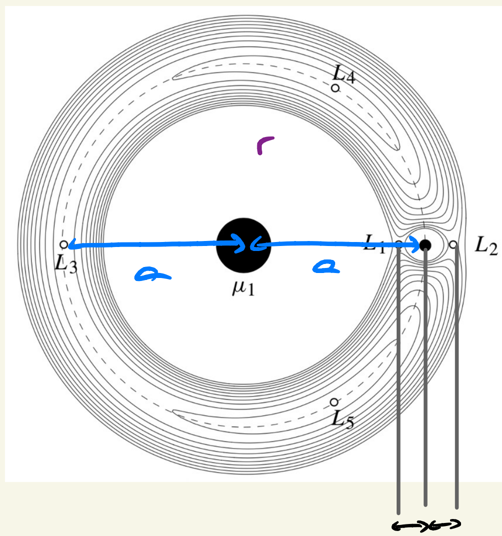
# COLINEAR LAGRANGIAN POINTS

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)\frac{(x+\mu_2)}{r_1}+\mu_2\left(-\frac{1}{r_2^2}+r_2\right)\frac{(x-\mu_1)}{r_2}=0$$

$$\cancel{\mu_1\left(-\frac{1}{r_1^2}+r_1\right)\frac{y}{r_1}}+\cancel{\mu_2\left(-\frac{1}{r_2^2}+r_2\right)\frac{y}{r_2}}=0$$

$$y=0$$




$$\Delta = R_H = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3}$$

$$r_1 = x+\mu_2$$

$$+\; r_2 = \mu_1 - x$$

$$r_1+r_2 = 1$$

$$\frac{\partial r_1}{\partial x}=1$$

$$\frac{\partial r_2}{\partial x}=-1$$

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)\underbrace{\cancel{\frac{(x+\mu_2)}{r_1}}}_{1}+\mu_2\left(-\frac{1}{r_2^2}+r_2\right)\underbrace{\cancel{\frac{(x-\mu_1)}{r_2}}}_{-1}=0$$

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)-\mu_2\left(-\frac{1}{r_2^2}+r_2\right)=0$$

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)-\mu_2\left(-\frac{1}{r_2^2}+r_2\right)=0$$

$$r_1+r_2=1 \quad \therefore \quad r_1=1-r_2$$

$$\mu_1\left(-\frac{1}{(1-r_2)^2}+(1-r_2)\right)=\mu_2\left(-\frac{1}{r_2^2}+r_2\right)$$

$$\mu_1\left(\frac{\cancel{1}-(1-r_2)^3}{(1-r_2)^2}\right)=\mu_2\left(\frac{1-r_2^3}{r_2^2}\right)$$

$$(1-x)^3=\cancel{1}-3x+3x^2-x^3$$

$$\mu_1\left(\frac{3r_2-3r_2^2+r_2^3}{(1-r_2)^2}\right)=\mu_2\left(\frac{1-r_2^3}{r_2^2}\right)$$

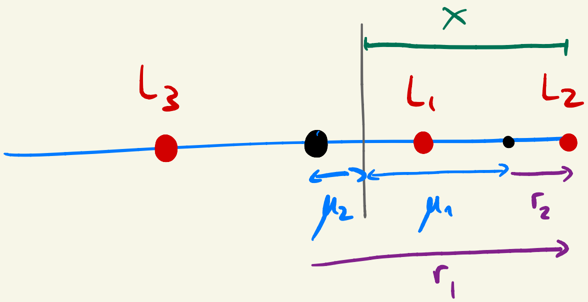
$$\mu_1\left(3r_2-3r_2^2+r_2^3\right)r_2^2=\mu_2(1-r_2)^2(1-r_2^3)$$

$$\mu_1 3r_2^3\left(1-\cancel{r_2}+\cancel{r_2^2/3}\right)=\mu_2(1-\cancel{r_2})^2(1-\cancel{r_2^3}) \qquad r_2 \ll 1$$

$$\boxed{r_2=\left(\frac{\mu_2}{3\mu_1}\right)^{1/3}}$$

HILL RADIUS

$L_1$



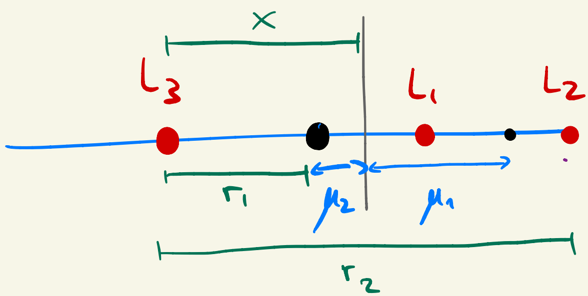

$$r_1 = x + \mu_2 \qquad \frac{\partial r_1}{\partial x} = 1$$

$$-r_2 = -x + \mu_1 \qquad \frac{\partial r_2}{\partial x} = 1$$

$$r_1 - r_2 = 1$$

$$\mu_1\left(-\frac{1}{r_1^2} + r_1\right)\cancelto{1}{\frac{(x+\mu_2)}{r_1}} + \mu_2\left(-\frac{1}{r_2^2} + r_2\right)\cancelto{1}{\frac{(x-\mu_1)}{r_2}} = 0$$

$$r_2 = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3} \qquad L_2$$




$$r_1 = -x - \mu_2$$

$$r_2 = -x + \mu_1$$

$$\frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = -1$$

$$r_2 - r_1 = 1 \qquad r_2 = 1 + r_1$$

$$\mu_1\left(-\frac{1}{r_1^2} + r_1\right)\cancelto{-1}{\frac{(x+\mu_2)}{r_1}} + \mu_2\left(-\frac{1}{r_2^2} + r_2\right)\cancelto{-1}{\frac{(x-\mu_1)}{r_2}} = 0$$

$$\mu_1\left(-\frac{1}{r_1^2} + r_1\right) + \mu_2\left(-\frac{1}{r_2^2} + r_2\right) = 0$$

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)+\mu_2\left(-\frac{1}{(1+r_1)^2}+1+r_1\right)=0$$

$$\frac{\mu_2}{\mu_1}=\frac{(1-r_1^3)(1+r_1)^2}{r_1^3(r_1^2+3r_1+3)}$$

$r_1 = 1+\beta$

$$\frac{\mu_2}{\mu_1}=\frac{\beta^2\left(1-(1+\beta)^3\right)}{(1+\beta)^3\left[(1+\beta)^2+3(1+\beta)+3\right]}$$

$$=\frac{\beta^2\left[\cancel{1}-(\cancel{1}+3\beta^2+3\beta+\beta^3)\right]}{(1+\beta)^3\left[(1+\beta)^2+3(1+\beta)+3\right]}$$

$$=\beta^3\frac{\left[3+\cancel{3\beta}+\cancel{\beta^2}\right]}{(1+\cancel{\beta})^3\left[3+3(1+\cancel{\beta})+(1+\cancel{\beta})^2\right]}$$

$\beta \ll 1$

$$\frac{\mu_2}{\mu_1}\approx\beta^3 \quad\therefore\quad \beta\approx\left(\frac{\mu_2}{\mu_1}\right)^{1/3}\approx 10^{-1}\ll 1$$

$$\boxed{r_1\approx 1}$$