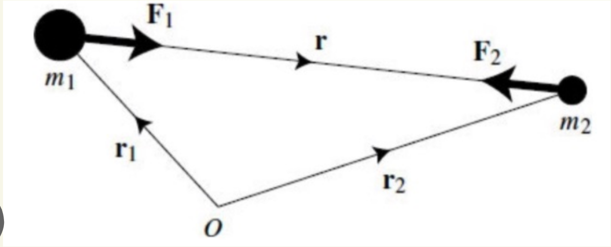
2-Body Problem

$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$



$$\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu \mathbf{r}}{r^3} = 0 \quad (\times \mathbf{r})$$

$$\mu = G(m_1 + m_2)$$

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0 \longrightarrow \boxed{\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}}$$

$$\mathbf{r} = r\hat{\mathbf{r}}$$

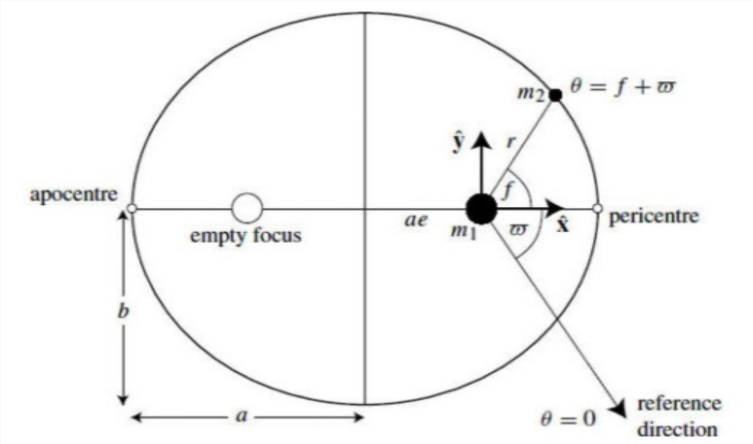
$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\mathbf{h} = r^2\dot{\theta}\hat{\mathbf{z}}$$

$$r = \frac{p}{1 + e\cos(\theta - \bar{\omega})}$$

$$p = h^2/\mu$$

GEOMETRY	ECCENTRICITY	SEMILATUS RECTUM $p = h^2/\mu$	ENERGY
CIRCLE	$e = 0$	$p = a$	$E = E_{min}$
ELLIPSE	$0 < e < 1$	$p = a(1-e^2)$	$E_{min} < E < 0$
PARABOLA	$e = 1$	$p = 2q$	$E = 0$
HYPERBOLA	$e > 1$	$p = a(e^2-1)$	$E > 0$



$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\varpi)}$$

$$f = \theta - \varpi$$

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

$$x = r\cos f$$

$$y = r\sin f$$

$$T^2 = \frac{4\pi^2}{\mu} a^3$$

$$n \equiv \frac{2\pi}{T} \quad \text{MEAN MOTION}$$

$$\mu = n^2 a^3$$

$$h^2 = p \cdot \mu = a(1-e^2)\mu$$

$$= n^2 a^4 (1-e^2)$$

$$h = n a^2 \sqrt{1-e^2}$$

$$\ddot{r} + \frac{\mu r}{r^3} = 0 \quad (\cdot \dot{r})$$

$$\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\mu}{r^3}\dot{\mathbf{r}}\cdot\mathbf{r} = 0 \qquad \begin{array}{l} \mathbf{r} = r\hat{r} \\ \dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \end{array} \rightarrow \dot{\mathbf{r}}\cdot\mathbf{r} = \dot{r}r$$

$$\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\mu}{r^2}\dot{r} = 0$$

$$\frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - \frac{\mu}{r} = C \qquad \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = v^2$$

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C \qquad \boxed{C, h}$$

$$\dot{\theta} = \frac{d}{dt}(f + \bar{\omega}) = \dot{f}$$

$$v^2 = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \dot{r}^2 + r^2\dot{f}^2$$

$$r = \frac{a(1-e^2)}{1+e\cos f} \rightarrow \dot{r} = \frac{r\dot{f}e\sin f}{1+e\cos f}$$

$$r^2\dot{f} = h = na^2\sqrt{1-e^2}$$

$$\dot{r} = \frac{r^2\dot{f}e\sin f}{r(1+e\cos f)} = \frac{na^2\sqrt{1-e^2}\,e\sin f}{r(1+e\cos f)} = \frac{na}{\sqrt{1-e^2}}e\sin f$$

$$\boxed{\dot{r} = \frac{na}{\sqrt{1-e^2}} e \sin f}$$

$$v^2 = \dot{r}^2 + r^2 \dot{f}^2$$

$$r^2 \dot{f} = h = na^2 \sqrt{1-e^2}$$

$$\boxed{r\dot{f} = \frac{h}{r} = \frac{na}{\sqrt{1-e^2}} (1 + e\cos f)}$$

$$v^2 = \frac{n^2 a^2}{(1-e^2)} \left[e^2 \sin^2 f + 1 + e^2 \cos^2 f + 2e\cos f \right]$$

$$= \frac{n^2 a^2}{(1-e^2)} \left[1 + e^2 + 2e\cos f \right]$$

$$r = \frac{a(1-e^2)}{1+e\cos f} \longrightarrow 1 + e\cos f = \frac{a}{r}(1-e^2)$$

$$v^2 = \frac{n^2 a^2}{(1-e^2)} \left[2(1+e\cos f) - (1-e^2) \right]$$

$$= \frac{n^2 a^2}{\cancel{(1-e^2)}} \left[2\frac{a}{r}\cancel{(1-e^2)} - \cancel{(1-e^2)} \right]$$

$$v^2 = n^2 a^2 \left[2\frac{a}{r} - 1 \right] \times \frac{a}{a} \qquad \mu = n^2 a^3$$

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C$$

$$\mu\left(\frac{1}{\cancel{r}} - \frac{1}{2a}\right) - \cancel{\frac{\mu}{r}} = C \qquad \boxed{C = -\frac{\mu}{2a}}$$