Foundations of the Special Theory of Relativity

Einstein: *"On the Electrodynamics of Moving Bodies" (1905)*

The Demise of the Luminiferous Æther

• The Propagation of Waves (Maxwell 1861)
.... but Maxwell's Equations "fail" under Galilean transformation

• Properties required of the luminiferous æther
..... rigidity, dissipationless, permeability, ...
Is a preferred frame consistent with the Principle of Relativity?
.... frustrated by the Michelson-Morley Experiments 1897

Salvage Attempts
Fitzgerald- Lorentz Contraction: $L_v = L_o[1 - (v/c)^2]^{+1/2}$
... accompanied by "Ether Drag" in a "boundary layer"
.... frustrated by Fizeau's experiments and the aberration of starlight

The Foundations of Special Relativity

(Two Postulates)

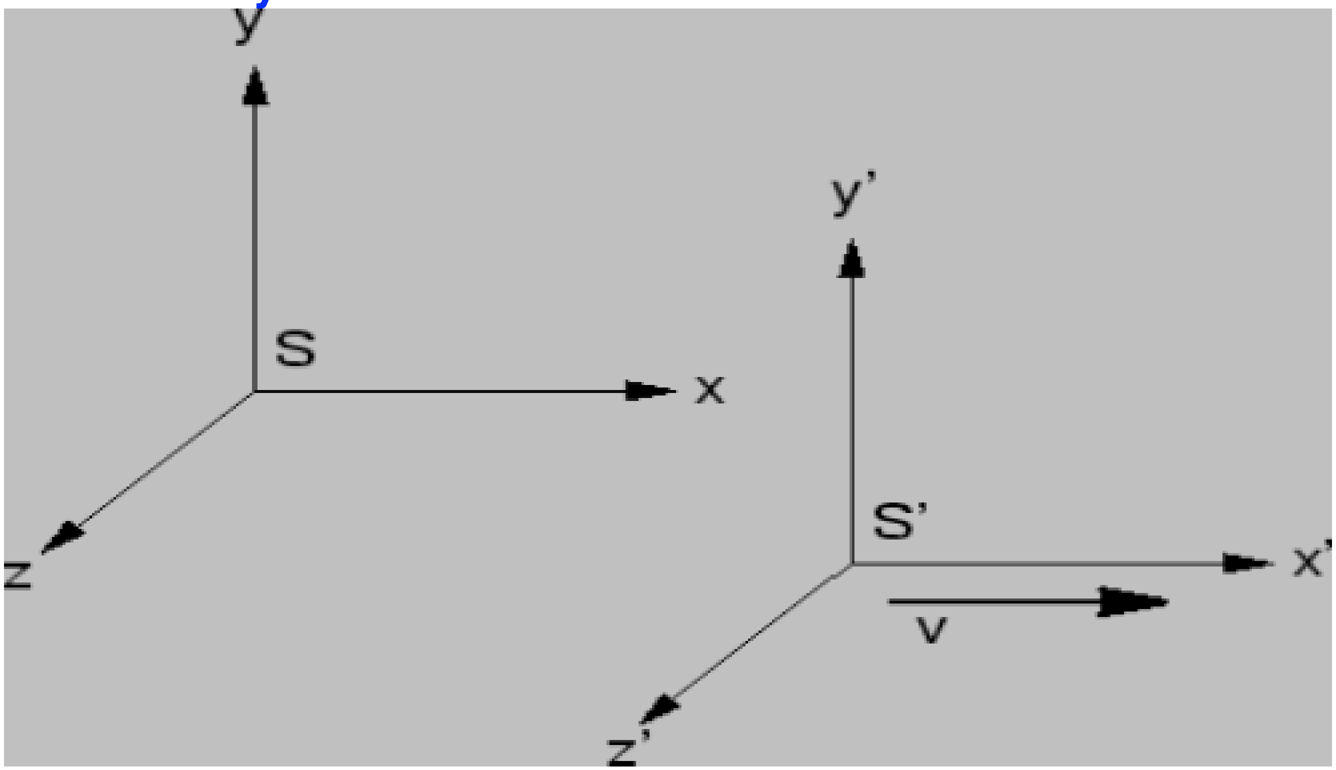
• The Principle of Relativity (Galileo 1632)

• The Observed Constancy of the Speed of Light, c

Special Relativity & The Lorentz Transformation

Setup

- Cartesian Systems: S = (x,y,z,t) and S' = (x',y',z',t')
- Relative motion is along the +x and -x' axes with <u>constant</u> speed v
- Systems coincide S = S' when t = t' = 0



Objective

Find a coordinate transformation that (1) satisfies the Principle of Relativity <u>and</u> (2) preserves the constancy of the speed of light.

The Lorentz Transformation

<u>The Galilean Transformation</u>

$x = x' + v\,t$, $y = y'$, $z = z'$ and $t = t'$

...giving **u** = **u'** + **v**, for example, which will <u>not</u> satisfy the second postulate.

<u>Construction (and a bit of algebra)</u>

A pulse of light is emitted at $(x,y,z,t) = (x',y',z',t') = (0,0,0,0)$

i.e., when the origins coincide at $t = t' = 0$

At later times the wave fronts are <u>both</u> spherical and described by:

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

and

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

as required by the second postulate

Assuming a linear transformation (which is required by an isotropic and homogeneous space) the <u>only</u> possible transformation is a scale change s(v):

$$(x^2 + y^2 + z^2 - c^2t^2) = s(v)\,(x'^2 + y'^2 + z'^2 - c^2t'^2)$$

- **Consider a transformation from S to S' - and then back to S from S'**
  This requires $s(v)s(-v) = 1$ for all v

- **Any value for s(v) implies a scale change in directions orthogonal to v. Symmetry then requires that $s(v) = s(-v)$. Hence**
  $s(v) = 1$ for all v

... which is about as simple as it gets!

The Lorentz Transformation

• With $s(v) = 1$ for all v, and v along x and x', symmetry also requires

$$y = y' \text{ and } z = z'$$

so we need only to find transformations satisfying

$$(x^2 - c^2t^2) = (x'^2 - c^2t'^2)$$

<u>The most general linear transformations for the remaining two coordinates are</u>

$$x' = a_1x + a_2t$$

and

$$t' = b_1t + b_2x$$

where the coefficients are functions of v only.

<u>Obviously</u>

$$a_1(0) = b_1(0) = 1 \text{ and } a_2(0) = b_2(0) = 0$$

The velocity of the origin of S' as seen in S is v so its value is $x = vt$ so

$$a_2 = -va_1$$

<u>In any case, solving for the coefficients gives</u>

$$a_1 = b_1 = \gamma \text{ ,}$$

$$a_2 = -v\gamma \text{ and } b_2 = \gamma v/c^2$$

where γ is the Lorentz Factor $\gamma = [1 - (v/c)^2]^{-1/2}$

The Lorentz Transformation

Finally, we have

$$x' = \gamma(x - vt)$$
$$y' = y,\ z' = z$$

and

$$t' = \gamma(t - vx/c^2)$$

where

$$\gamma = [1 - (v/c)^2]^{-1/2}$$

The x-transformation is just that proposed by Lorentz to "fix" the æther problem. However, the time coordinate also transforms in a non-Galilean fashion.

Time is no longer absolute!

The observed time interval between events will depend upon the state of motion of the observer

However the temporal ordering of events, corrected for travel time delays, will not!

(This is a very good thing.)

Emphasis

The Lorentz Transformation follows from the postulates of Special Relativity

Bonus: Maxwell's Equations are invariant under a Lorentz Transformation!

Mass, Energy, and Momentum in Special Relativity

The Newtonian definition of force is adopted

$\mathbf{F} = d\mathbf{p}/dt$

Momentum is conserved independent of coordinate system which requires

$\mathbf{p} = \gamma m_o \mathbf{v}$

where is the inertial mass as measured in a co-moving (v = 0) system.

This also requires for energy-conserving situations that the energy be given by

$E = \gamma m_o c^2$

The Newtonian Limit

For velocities **v** small compared to the speed of light:

- The Lorentz Factor approaches unity
- The Lorentz Transformation becomes Galilean
- The equations of motion become Newtonian

except that

$$\mathbf{E = \gamma m_o c^2 = m_o c^2 + mv^2/2 + ...}$$

so a material particle at rest possesses a rest mass energy $E_o = m_o c^2$

Summary: The Special Theory of Relativity

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Principal Predictions and Consequences

Testing the Postulates

- **The constancy of the speed of light has been repeatedly tested and verified.**
- **The Principle of Relativity has been vigorously tested indirectly.**

Length Contraction: $\Delta L = \gamma^{-1}\, \Delta L'$

where $\gamma = [1 - (v/c)^2]^{-1/2} \geq 1$ is the Lorentz Factor

Time Dilation: $\Delta t = \gamma^{-1}\, \Delta t'$

Note that $\Delta L/\Delta t = \Delta L'/\Delta t'$, ensuring constancy of c, but simultaneity is lost!

Relativity of Inertial Mass and Momentum

Inertial mass: $m = \gamma m_o$ where m_o is the rest mass

Momentum: $\mathbf{p} = m\mathbf{v} = \gamma m_o \mathbf{v}$

The Equivalence of Mass and Energy

Total Energy: $E = mc^2$ or $E = \gamma m_o c^2$

Rest Energy: $E_o = m_o c^2$

The Special Theory of Relativity: Some other relations

"Kinetic Energy"

$K = E - E_o = (\gamma - 1)m_o c^2$

.... *versus* the Newtonian $K = mv^2/2$

The 4-Momentum

$E^2 = (pc)^2 + (m_o c^2)^2$

.... *versus* the 3-momentum $p = mv$

Addition of (Parallel)Velocities

$u = (u' + v)/(1 + u'v/c^2)$

.... *versus* $u = u' + v$)

Here v is the velocity of the S' system relative to S

Note that this ensures $u \leq c$ always.

The Transverse Doppler Effect

....$\lambda_{observed} = \gamma\lambda_{emitted}(1 + v_{radial}/c)$

... *versus* $\lambda_{observed} = \lambda_{emitted}(1 + v_{radial}/c)$

Note that $\lambda_{observed} = \gamma\lambda_{emitted}$ when $v_{radial} = 0$

Digression: Newton's Second Law of Mechanics

Newton and Einstein: $\mathbf{F} = d\mathbf{p}/dt$ where $\mathbf{p} = m\mathbf{v}$

The form $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a} = d\mathbf{v}/dt$ is only true if m is a constant!

This is not the case in Newtonian mechanics when $m = m(t)$

It is never the case in relativistic mechanics where $m = m(v) = \gamma m_o$

In Special Relativistic mechanics:

$$\mathbf{F} = m[\mathbf{a} + (\gamma/c)^2(\mathbf{v}\cdot\mathbf{a})\mathbf{v}]$$

...and the acceleration is generally not parallel to the force!

Limitations of Special Relativity

The precepts of Special Relativity apply to inertial (unaccelerated) systems.

That is, v must be a constant velocity

The spacetime geometry must be locally flat (Minkowskian)