Aristarchus

Aristarchus measured the angle between the quarter Moon and the Sun as 1/120 of a full circle or 3°. The length of a 3° arc is 1/120 of the circumference which is 2π times the radius. The radius in this instance is the Sun's distance d_{Sun} and the arc length is the Moon's distance d_{Moon} so we have (approximately) $\alpha = d_{Sun}/d_{Moon} = 120/2\pi = 19$. (Aristarchus actually estimated that $18 \leq \alpha \leq 20$.) We then have $\boldsymbol{\alpha = d_{Sun}/d_{Moon} = 19}$

The Moon and Sun have the same apparent (angular diameters) as is shown by the almost exact coverage of the Sun by the Moon during a solar eclipse. We cvan conclude that their diameters, D, are in the same ratio as their distances: We have $\boldsymbol{\alpha = D_{Sun}/D_{Moon} = 19}$

Aristyarchus estimated the angular diameters of both theSun annd the Moon to be about 1/180 of a circle or 2°. That implies (see above) $d_{Sun}/D_{Sun} = d_{Moon}/D_{Moon} = 180/2\alpha = 28.6$ so we will adopt $\boldsymbol{\beta = d_{Sun}/D_{Sun} = d_{Moon}/D_{Moon} = 29}$.

Now we come to thhe more complicated part, putting these numbers in units of the Earth's diameter $\mathbf{D_{Earth}}$. Let $\mathbf{x = D_{Moon}/D_{Earth}}$. Given the value of **x** we can express the diameter of the Sun as well as the distances of both the Sun and Moon in terms of the Earth's diameter. Using the above numbers, fo example,

$$\mathbf{D_{Moon}/D_{Earth} = x,\ D_{Sun}/D_{Earth} = \alpha x,\ d_{Moon}/D_{Earth} = \beta x,\ and\ d_{Sun}/D_{Earth} = \alpha\beta x}$$

So what is X? Aristarchus observed that during a lunar eclipse the diameter of Earth's shadow (at the distance of the Moon) appeared to be about twice the apparent size of the Moon (i.e., about 4° by Aristarchus' estimate). Letting D_S be the diameter of that shadow we have $\boldsymbol{\gamma = D_S/ D_{Moon} = 2}$.

Note that the values of α, β and γ are all determined from observations.

Let d_S be the distance from the Earth of the tip of the Earth's umbral shadow. By similar triangles we have:

$$\mathbf{d_S/ D_{Earth} = (d_S - d_{Moon})/ D_S = (d_S + d_{Sun})/ D_{Sun}}$$

Eliminate D_S by replacing it with $\mathbf{D_S = \gamma D_{Moon}}$ in this expressionn. That gives:

$$\mathbf{d_S/ D_{Earth} = (d_S - d_{Moon})/ \gamma D_{Moon} = (d_S + d_{Sun})/ D_{Sun}}$$

But $\mathbf{D_{Sun} = \alpha D_{Moon}}$ and $\mathbf{d_{Sun} = \alpha d_{Moon}}$, so this last expression can be written as:

$$\mathbf{d_S/ D_{Earth} = (d_S - d_{Moon})/ \gamma D_{Moon} = (d_S + \alpha d_{Moon})/\alpha D_{Moon}}$$

Letting $\mathbf{D_{Moon}/D_{Earth} = x}$ and multiplying through by $\mathbf{D_{Moon}}$ gives

$$\mathbf{x d_S = (d_S - d_{Moon})/ \gamma = (d_S + \alpha d_{Moon})/\alpha}$$

The first equality gives $\mathbf{d_S = d_{Moon}/ (1- \gamma x)}$ and inserting this into the second equality, and dividing through by $\mathbf{d_{Moon}}$ and solving for **x** gives the desired result

$$\mathbf{x = (1 + \alpha)/\alpha(1 + \gamma)}$$

Using Aristarchus' "observed"values of $\alpha = d_{Sun}/d_{Moon} = 19$ and $\gamma = D_S/ D_{Moon} = 2$ gives

$D_{Moon}/D_{Earth} = x = 20/57$ (about 1/3)

$d_{Moon}/D_{Earth} = \beta x = 580/57$ (about 10)

$D_{Sun}/D_{Earth} = \alpha x = 19/3$ (about 6)

$d_{Moon}/D_{Earth} = \alpha\beta x = 11{,}020/57$ (about 190)

Check Aristarchus's actual adopted values. 87°, 2° giving (18-20), (19/3-43/6)

3° - 1/30 quadrant, 2° = 1/15 of zodiacal sign (30°)