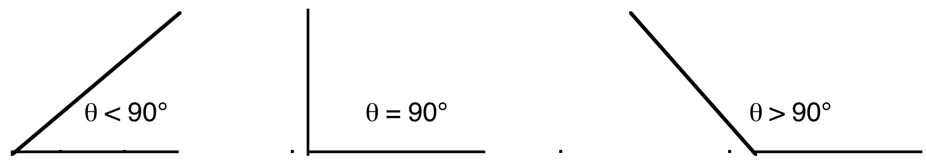
Angles and Angular Measurement

Angles and Angular Units

An angle measures the difference between two directions just as a length describes the spatial distance between locations. Just as various units (feet, meters, light-years) can be used to describe length or distance, so different units can be used to measure angle. The mathematicians and philosophers of classical times preferred to express angles as fractions of the full circle, with values lying between zero and one. Today we usually use a system originating with the ancient Babylonians in which the full arc of the circle is divided into 360 equal parts called *degrees* (°). The origin of this system is astronomical, one degree (1°) is approximately equal to the daily motion of the Sun among the stars of the celestial sphere. Mathematicians prefer to represent the full circle by $2\pi = 6.283185...$ *radians* where π (“pi”) is the ratio of a circle’s circumference to its diameter*. Thus 1 radian is about equal to 57 degrees.

Angular Nomenclature

Two opposite direction (North and South, Zenith and Nadir, for examples) are separated in angle by a half circle ($\theta = 1/2$), 180 degrees ($\theta = 180°$), or π radians ($\theta = \pi$). A pair of perpendicular directions (North and East, Vertical and Horizontal, ..) are said to form a right angle corresponding to a quarter circle, 90°, or $\pi/2$ radians. Angles less than a right angle are said to be acute, those greater than 90° are obtuse. Notice that two lines which make an angle θ with respect to one another also make an angle of $360° - \theta$ with respect to each other; it just depends upon the direction in which one chooses to measure things:



An acute angle (about 40°) | A right angle (exactly 90°) | An obtuse angle (about 130°)

Very small angles can be expressed as small fractions or as decimal parts of the basic unit. One can, for example, speak of thousandths of a degree. Historically, however, the degree (1° = 1/360 th circle) was subdivided into 60 parts called *minutes of arc* (‘) or simply *minutes* (1° = 60’, 1’ = 1°/60) and each minute divided into 60 *seconds of arc* (“) or *seconds* (1’ = 60”, 1”‘=1’/60). Thus

1 full circle = 360° = 21,600’ = 1,296,000”

A useful conversion number to remember for astronomical purposes is that 1 radian = 206,265”.

Measuring Angles: Some Examples

A *protractor* is a circular disk with graduated markings. It is useful for measuring angles on a drawing. A surveyor’s *transit* or a mariner’s *sextant* is basically a precision protractor equipped with a telescope or other sighting device for more accurate measurements. The human body is the basis for some units of length measurement (*e.g.* , the yard) and can also be used to measure angles in an approximate way: The span of your open hand at arm’s length is roughly 20° and the width of your palm about 8° whereas the width of your thumb subtends about 2.5°. It is instructive to “calibrate” your body parts against something of known angular size: How many Moon diameters are covered by your big toe? How many “hand widths” does it take to circle the 360° horizon. Note that, on the sky, the angular diameter of the Sun or Moon is close to 1/2° and both appear to move about 15° per hour because of the Earth’s rotation. The “bowl” of the Big Dipper is about 6° across.

*There are more obscure units: The traditional mariner’s compass divides the circle into 32 named *points* of the compass. (Examples: north, south-southwest, north by northwest,..). Artillery men divide the circle into 6400 *mils* for purposes of aiming their guns. Just to add to the confusion, there is an “infantry mil” which is equal to 1.018 “artillery mils” . Neither, of course, should be confused with the length unit (1 mil = 0.001 inch) nor the monetary unit (1 mil = 0.001 Israeli pound) nor with the abbreviation for milliliter which is a liquid volume of about one cubic centimeter......

Angular Size

The apparent size of an object depends both upon its physical dimension and its distance; all other things being equal, more distant objects will look smaller. The apparent size can be measured in terms of an *angular size* expressed as an angle α. Note that the apparent size can also depend on the orientation of the object; a snake looks smaller "face on" than "sideways". However, many astronomical objects are spherical, or nearly so (stars, planets,..) and spheres present the same circular aspect when viewed from any direction. The angular sizes of things astronomical are usually very small. In terms of angular sizes, the Sun and Moon are usually the largest astronomical objects seen on the sky and they are only about 1/2° in angular diameter. The fact that an angular size is small means that an object's physical diameter, **D**, is much smaller than its physical distance, **d**. Indeed, both the Sun and the Moon are at distance about 115 times their respective diameters. For angles this small, or smaller, the relationship between angular size, physical size, and distance is quite accurately given by

$$\mathbf{D} = \mathbf{d} \times \alpha(\text{radians})$$

where α is in radians and **d** and **D** are given in the same length units. If (as is common in astronomy) the angle is measured in arc-seconds (") we need to use the conversion factor:

$$\mathbf{D} = \mathbf{d} \times \alpha('') \div 206{,}265$$

where **d** and **D** are again in the same units. Sometimes "mixed" units are used, which requires additional conversion factors. For example, **D** , might represent the distance between two stars in a binary system at a distance **d** from the Earth. Orbital dimensions are commonly expressed in *astronomical units* (au) which is the average Earth-Sun distance. Interstellar distances are commonly expressed in *parsecs*.(pc). The parsec (pc) is defined to be equal to 206,265 astronomical units (surprise!) so that **D**(au) **=** **d**(pc) x α(").

Example: The planet Jupiter, when closest to the Earth, is at a distance of 4.2 astronomical units or about **d** = 6.3×10^9 kilometers. At that time its angular diameter is about α = 23". Its physical diameter is, therefore **D = d** x α(") ÷ 206,265 = 70,000 kilometers.

Example: Two stars in a binary system are observed to be separated by 3" on the sky. The system's distance is 8 parsecs. The two stars are therefore at least 8 x 3 = 24 au apart . (Why "at least"?)

Example: The inhabitants of the α Centauri system at a distance of 1.35 parsecs (see below) look at our system and detect the planet Jupiter which orbits the Sun at a distance of 5.20 astronomical units. The biggest angle they will observe between Jupiter and the Sun will be α(") = 5.20 (au)÷1.35 (pc) = 3.85"

Trigonometric Parallaxes

Because of then Earth's annual motion about the Sun stars on the sky will appear to move in tiny elliptical paths. (This is the parallactic motion that Aristotle was unable to see.) Indeed, the apparent path of a star will be identical in apparent (angular) size and shape to the Earth's annual orbit as seen from the star. While the shape of this ellipse will depend upon the star's direction, the large dimension of the apparent ellipse (*i.e.,* twice its semimajor axis) will depend only upon the star's distance. Recollect that the Earth's orbital dimension is (**D** = 1 astronomical unit) so that, if we express the star's distance, **d,** in parsecs and the parallax angle **p** in arc-seconds we have **p** = α = **D/d** or **d**(parsecs) = 1/**p**("). The nearest star (after the Sun) shows a parallactic ellipse of semimajor axis **p** = 0.74" so its distance is **d** = 1.35 parsecs or about 278,000 astronomical units. (An astronomical unit is about 150 million kilometers, a parsec is about 31 trillion kilometers or 3.26 light-years.)

A Bit of Trigonometry

Consider a triangle with a right (90°) angle and with sides of lengths A, B, and C where side C is the side opposite the right angle (this longest side is called the "hypotenuse") . Let the angle opposite side A be **a**, that opposite side B be **b**, and that opposite C be **c** = 90°. (By the way, the sum of the three angles of a plane triangle is always 180°.) The basic trigonometric functions of an angle α are the sine (sinα), the cosine (cosα) and the tangent (tan α). In terms of the sides of our particular right triangle, sin**a** **=** cos**b** = A/C , cos**a** = sin**b** = B/C , tan**a** = A/B = 1/ tan**b**. For very small angles α it is *approximately* true that cosα = 1 and that sinα **=** α and tanα = α , so long as α is measured in radians. (Remember, a radian is about 57°.)