An Algebra Primer

Equations

One way of looking at algebra is as a simple set of rules for manipulating symbols in equations. The symbols usually represent (unspecified) numbers which, in turn, describe the quantities of "things".
An equation is just a statement of some sort of relationship between these quantities. The most common form of equation is an equality which uses the symbol "=" to indicate that two quantities represented by the symbols **A** and **B**, say, are numerically the same:

$$\mathbf{A} = \mathbf{B}$$

If **A** represents the number of males and **B** the number of females, this just states that the two <u>numbers</u> (whatever they are) are the same, a satisfactory state of affairs. This is useful; if you know how many females there are the equation enables you to "calculate" the number of males. The reverse is also true since, if $\mathbf{A} = \mathbf{B}$ then $\mathbf{B} = \mathbf{A}$. That seems simple enough.

Other Relations

Equations can describe relations other than equality (=) and one uses different symbols to represent those other relationships. Some examples are:

A is not numerically the same as **B:**	$\mathbf{A} \neq \mathbf{B}$	$(8 \neq 14)$
A is numerically larger than **B**:	$\mathbf{A} > \mathbf{B}$	$(14 > 8)$
A is numerically smaller than **B:**	$\mathbf{A} < \mathbf{B}$	$(8 < 14)$
A and **B** are roughly equal :	$\mathbf{A} \sim \mathbf{B}$	$(14 \sim 15)$

and others such as "very much greater than" (»), "less than or equal to" (≤), "almost the same as" (≈), and so forth. For the present, we shall confine our attention to the equality (=) relationship only.

Operations

There are all sorts of operations one can perform with symbols other than just stating a relationship between them. The most familiar ones are those of ordinary arithmetic; addition (**+**), subtraction (**-**), multiplication (**x** or **•**), and division (**/** or **÷**). To use the above example, $\mathbf{A} + \mathbf{B}$ is the number of males and females, $\mathbf{A} - \mathbf{B}$ is the number of superfluous males (hopefully a negative number), $\mathbf{A}/\mathbf{B} = \mathbf{A} \div \mathbf{B}$ is the ratio of males to females, and I'm afraid to guess what $\mathbf{A} \times \mathbf{B}$ would represent. (Actually, it's the number of possible male-female pairings.) The "**x**" of multiplication is often omitted: **AB** is the same as $\mathbf{A} \times \mathbf{B}$. There are other operations such as "squaring" or "taking the cube root of" which we'll worry about later.

For some operations the order in which you do them doesn't make any difference:
Addition and multiplication are examples: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ and $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$
But for others, like subtraction or division: $(\mathbf{A} - \mathbf{B}) \neq (\mathbf{B} - \mathbf{A})$ and $\mathbf{A} \div \mathbf{B} \neq \mathbf{B} \div \mathbf{A}$, except in special cases.
[Can you figure out which are the "special cases" here?]

Rules for Manipulating Equations

Suppose we have an equation which describes an equality (*i.e.,* is "true"): $\mathbf{A} = \mathbf{B}$
If we add (or subtract) the same quantity to (or from) from both sides of the equation, the resulting equation is still true:

If $\mathbf{A} = \mathbf{B}$, then $\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C}$ and $\mathbf{A} - \mathbf{C} = \mathbf{B} - \mathbf{C}$ for any quantity **C**

The same is true for multiplication (or division) by a quantity **C**:

If $\mathbf{A} = \mathbf{B}$, then $\mathbf{A} \times \mathbf{C} = \mathbf{B} \times \mathbf{C}$ and $\mathbf{A} \div \mathbf{C} = \mathbf{B} \div \mathbf{C}$ for any quantity **C**

.....except that division by $\mathbf{C} = 0$ is not allowed. (Division by zero usually gives an infinitely large positive or negative number; your calculator expresses its unhappiness by beeping, flashing or bursting into flame.)
Basically, you can do almost anything to an equation involving an equality, so long as you do the same thing to both sides, and, if it was true when you started, it'll be true when you are finished.
[Think about this: If the relation is "≠" instead of "=", are the above rules still correct? What about ">" ?]

Special Quantities

There are two very special values that can be assumed by some quantities, namely zero (**0**) and one (**1**).

If $\mathbf{A} = \mathbf{B}$, then $\mathbf{A} / \mathbf{B} = \mathbf{1}$ and $\mathbf{A} - \mathbf{B} = \mathbf{0}$

In particular $\mathbf{A} \div \mathbf{A} = \mathbf{1}$ and $\mathbf{A} - \mathbf{A} = \mathbf{0}$ for any quantity **A**

Also $\mathbf{A} \div \mathbf{1} = \mathbf{A} \times \mathbf{1} = \mathbf{A}$ $\quad \mathbf{A} + \mathbf{0} = \mathbf{A}$ $\quad \mathbf{A} - \mathbf{0} = \mathbf{A}$ and $\mathbf{A} \times \mathbf{0} = \mathbf{0}$

Solving Equations

Solving equations is sometimes (but not always) just a matter of rearrangement. In many cases of interest we have some sort of an equality relationship between three quantities, say **A**, **B**, and **C**, we know values (numbers) for two of them, and we want to figure out the third. Going back to our first example we might write **C** = **A** + **B** in which case **C** represents the number of males plus females. But, this equation can be manipulated following the above rules, to give all sorts of equivalent forms such as:

C = **A** + **B**	Both sides multiplied or divided by one, or zero added to or subtracted from both sides.
A = **C** - **B**	**B** subtracted from both sides, sides swapped
B = **C** - **A**	**A** subtracted from both sides, sides swapped
C - **A** - **B** = 0	**A**+**B** subtracted from both sides
C/(**A** + **B**) = 1	Both sides divided by (**A** + **B**) ....so long as (**A** + **B**) is not equal to zero

....and so forth. The point is that all of the above are really the same equation since they all say the same thing. No manipulation of an equation really provides any really "new" information. However, one form might be more useful than another in some particular case. Note: Parentheses indicate that the contents (the stuff between them) are to be treated as a single quantity. Thus **C**/(**A** + **B**) is not the same thing as **C**/**A** + **B** = (**C**/**A**) + **B** = **B** + (**C**/**A**). Overlooking this can get you into trouble.

Examples: 6/(2+4) = 1, but 6/2 + 4 = 7 and 6/4 + 2 = 3.5; (6 + 4)/2 = 5, but 6 + 4/2 = 8.

Powers and Roots

Suppose we want to multiply a number **A** by itself a number of times, say **n** times, to give another number **C**. Instead of writing **C** = **A**x**A**x....x**A** , where **A** appears **n** times, we use the shorthand notation:

$$\mathbf{C} = \mathbf{A}^{\mathbf{n}}$$

and say that "**C** equals **A** to the **n-**th power". The number **n** is the "exponent." It is also understood that

$$\mathbf{A}^{-\mathbf{n}} = 1/\mathbf{A}^{\mathbf{n}} \qquad \text{(and } 1/\mathbf{C} = \mathbf{C}^{-1}\text{)}.$$

It is useful to remember also that $\mathbf{A}^1 = \mathbf{A}$ and $\mathbf{A}^0 = 1$ for any number **A** (including zero).

Finally, $\mathbf{A}^{\mathbf{n}} \times \mathbf{A}^{\mathbf{m}} = \mathbf{A}^{\mathbf{m+n}}$ and $\mathbf{A}^{\mathbf{n}}/\mathbf{A}^{\mathbf{m}} = \mathbf{A}^{\mathbf{n-m}}$, etc.

(Note: Sums or differences of powers, like $\mathbf{A}^{\mathbf{n}} + \mathbf{A}^{\mathbf{m}}$, can't really be expressed in a more simple form.)

Sometimes one wants to know what number **A**, if raised to the **n-**th power, would give a known number **C**. That number **A** is called "the **n**-th root of **C**" and can be written as

$\mathbf{A}^{\mathbf{n}} = \mathbf{C}$ as above, or as $\mathbf{A} = \mathbf{C}^{\mathbf{1/n}}$, or sometimes as $\mathbf{A} = \sqrt[\mathbf{n}]{\mathbf{C}}$

(If no **n** is given in the last expression, **n** = 2 is assumed; just plain "$\sqrt{x}$" is "the square root of x"; *e.g.*, $\sqrt{9}=3$)

One can even have fractional powers: $\mathbf{A}^{\mathbf{a/b}} = (\mathbf{A}^{\mathbf{a}})^{\mathbf{1/b}} = (\mathbf{A}^{\mathbf{1/b}})^{\mathbf{a}}$

(Actually, the exponent can be a positive, negative, rational or irrational number, but we won't go into that.)

Some examples with numbers: $2^4 = 16$, $2^{-3} = 1/8$, $16^{1/2} = 4$, $10^5 = 100{,}000$, $2^{1/2} = \sqrt{2} = 1.414....$, etc.

Applications

There are many simple algebraic relations amongst astronomical quantities - and some not so simple. For example, the average density ρ of an object of mass **M** and volume **V** is given by ρ **= M/V**. If one knows any two of these quantities, one can calculate the third: **M =** ρ**V**, **V = M/**ρ, *etc.* Another important expression is "Kepler's Third Law" which relates the orbital size **a** (average separation) and the orbital period **P** of two objects with total mass **M** which are in orbit about one another :

$$\mathbf{a}^3 = \mathrm{A}\ \mathbf{M}\ \mathbf{P}^2$$

Here A is a numerical conversion factor whose value depends only upon what units one chooses for measuring **a** (feet, kilometers, light years?), **P**(seconds, fortnights, years?), and **M** (slugs, tons, grams?). Usually we will measure masses in units of the Sun's mass (M_o), orbital periods in years (yr), and orbital sizes in astronomical units (au) - the size of the Earth's orbit around the Sun. For the Earth-Sun system we have **a** =1 au, **P** = 1 yr, and **M** =1.000003 M_o (the Earth's mass is 0.000003 M_o). For these units the value of the constant A is given by A = (1x1x1)÷(1x1x1.00003) or A = 0.999997 . This is close enough to one for most purposes, so we'll set A = 1 and write $\mathbf{a}^3 = \mathbf{M}\mathbf{P}^2$**.** Now consider an asteroid with an orbital period **P** = 3 yr. Its mass is also tiny compared to the Sun, so **M** = 1 M_o. Its orbital size is then given by $\mathbf{a}^3 = 3^2 = 9$, or $\mathbf{a} = \sqrt[3]{9} = 1.817$ au. As another example, two stars found to be orbiting one another with a separation of **a** = 5 au and period **P** = 2 yr must have a total total mass $\mathbf{M} = \mathbf{a}^3/\mathbf{P}^2 = 5^3/2^2 = 125/4 = 31.25\ M_o$.