ASTR 565
Stellar Structure and Evolution
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New Mexico State University

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Contents

Part I

Stellar Structure

Unit 1

Equilibrium and time scales

1.1 Energy equilibrium

- Consider a thin layer at location r and of width dr in the nuclear burning region of a star. For energy equilibrium, the net flow of energy crossing this region's boundaries should be equal to the energy generation in the region.
- Let ε be the energy generated per gram of material per second, so the energy generated per second in (r, dr) is $\varepsilon \mathrm{d}m$.
- Consider the luminosity (energy per second) L that carries energy away. In the layer, there should be a balance between energy gains and energy losses (plus and minus denotes outgoing from center or toward center, respectively):

$$\begin{aligned} \varepsilon \mathrm{d}m &+ L_r^+ + L_{r+\mathrm{d}r}^- = L_r^- + L_{r+\mathrm{d}r}^+ &(1.1)\\ \varepsilon \mathrm{d}m &= [L_{r+\mathrm{d}r}^+ - L_{r+\mathrm{d}r}^-] - [L_r^+ - L_r^-] = L_{r+\mathrm{d}r} - L_r &(1.2)\\ \varepsilon \mathrm{d}m/\mathrm{d}r &= \mathrm{d}L/\mathrm{d}r, &(1.3)\end{aligned}$$

by dividing by dr and letting it go to zero (derivative).

- This standard relation will appear many times:

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V}, \tag{1.4}$$

or,

$$\mathrm{d}m = 4\pi r^2 \rho \, \mathrm{d}r \tag{1.5}$$

- If $\rho\varepsilon$ is the energy produced per second in each cm^3 of material, then by unit analysis the energy generated per second in (r, dr) is also given by

$$\varepsilon \mathrm{d}m = (4\pi r^2 \mathrm{d}r)\rho\varepsilon. \tag{1.6}$$

- Therefore, we can finally state

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \, \varepsilon. \tag{1.7}$$

- Sometimes we will use m instead of r as the independent coordinate, and so

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon. \tag{1.8}$$

- This is one of the main equations of stellar structure.
- Note that in regions where $\varepsilon = 0$, the luminosity is **constant**.
- The properties of ε, its temperature dependence and derivation, are discussed in Unit 3.
- See Problem 1.1.

1.2 Local thermodynamic equilibrium

- Collisions between particles in a gas and/or radiation allow equilibrium to occur if the distance particles travel and the time between collisions is small compared to macroscopic length and time scales.
- If this condition is met by radiation, it is known as blackbody radiation and the gas and radiation field are at the same temperature locally.
- This is known as *local thermodynamic equilibrium* (LTE).
- Specifically, the mean free path of photons is given by

$$\ell = \frac{1}{\kappa\rho}, \tag{1.9}$$

  where κ is the opacity and ρ is density.
- We'll see later that for a fully ionized gas that electron scattering gives $\kappa = 0.4\,\mathrm{cm}^2\,\mathrm{g}^{-1}$.
- Even for an average stellar density of $\rho = 1.4\,\mathrm{g\,cm}^{-3}$, the mean free path of photons is about 1 cm. Lots of collisions.
- In stellar interiors this is almost always the case. Above stellar photospheres this assumption breaks down.
- For example, the radiation leaving the Sun's surface is at about 6,000K. However, the gas (electron) temperature of the corona, through which this radiation passes, can be over 1,000,000K. The matter and radiation have not equilibrated. This is actually an outstanding problem.
- On the other end, the 6000K radiation from the Sun passes through Earth's 300K atmosphere also without (thankfully) equilibrating.
- In any case, assuming LTE allows us to calculate the interior structure of a star and all the thermodynamic properties in terms of temperature, density, and composition. This is done at each radial location, and as a function of time.

1.3 Are stars a one-fluid plasma?

- We know stars are mostly ionized. Shouldn't we treat the positive and negative charges as separately?
- If collisions are sufficiently frequent and Maxwellian and have times scales much less than other time scales of interest, then we can use a one-fluid model and generally ignore charge separation.
- The Debye length is the length over which an appreciable electric field can arise. It is roughly a measure of the thermal energy/electric potential energy ratio. If this length is short with respect to the plasma, we can assume charge neutrality.

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_\mathrm{B} T_e}{n_e q_e^2}} \approx 6.9 \left(\frac{T_e}{n_e}\right)^{1/2} \times 10\ \mathrm{m} \tag{1.10}$$

- In the core of the Sun, $n_e = 10^{32}$ m^{-3} and $T_e = 10^7$K, so that $\lambda_D \approx 10^{-11}$ m. In the photosphere, $n_e = 10^{11}$ m^{-3} and $T_e = 5 \times 10^3$ K, so that $\lambda_D \approx 1.5 \times 10^{-3}$ m. For the corona, we may find that $\lambda_D \approx 10$ m.
- So the mean free path of ions (electrons) are much smaller than the scale of the variations of physical quantities. Therefore we can treat most regions of stars as a one-fluid plasma system and use hydrodynamics. In coronae, however, it may be necessary to resort to *plasma physics* where many approximations are no longer valid.

1.4 Simple time scales of stars

While stars are for the most part static or quasistatic, and in equilibrium, there are time scales over which change may occur on a global scale. We can consider some quantity, say ϕ, and its rate of change $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$. Any relevant time scale is thus $\phi/\dot{\phi}$.

1.4.1 Dynamical timescale

- Let's first consider the smallest (and most observable) timescale $t_{\rm dyn}$.
- Consider a change in the fundamental dimension of the star, its radius $\phi = R$, may be possible to examine.
- Since gravity is the binding force, the velocity in a gravitational field is the escape velocity $\dot{\phi} = (2GM/R)^{-1/2}$.
- Then (neglecting factors of order unity),

$$t_{\rm dyn} = \left(\frac{R^3}{GM}\right)^{1/2}. \tag{1.11}$$

  Note that in terms of the mean density of a star $t_{\rm dyn} \approx (G\,\overline{\rho})^{-1/2}$.
- In terms of solar values, we find

$$t_{\rm dyn} \approx 30\,\mathrm{min}\left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M}{M_\odot}\right)^{-1/2}. \tag{1.12}$$

  So since we don't see large-scale changes on such time scales, we know there must be some balance of forces in the Sun. See Problem 1.2.
- Another way of thinking about this is to start with

$$g = \frac{GM}{R^2}. \tag{1.13}$$

- The time for a particle to fall under gravity is $\ell = 1/2gt^2 \rightarrow t = \sqrt{2\ell/g}$. The time scale therefore for a particle to fall, say, a distance $\ell = R/2$ in a star is

$$t_{\rm dyn} = \left(\frac{R^3}{GM}\right)^{1/2}, \tag{1.14}$$

  which reproduces Eq. (1.11).
- Dynamical processes in stars that help it adjust out of hydrostatic equilibrium typically occur over dynamical timescales, such as oscillations and even supernovae.

1.4.2 Thermal timescale

- There exist thermal processes that affect the internal energy of a star, which we'll denote as $\phi = U$.
- As we'll see from the Virial Theorem, $U \approx GM^2/R$.
- The energy changes due to radiation, whose rate of change is the luminosity $\dot{\phi} = L$.
- If we consider that losing its gravitational potential energy is the only real source of energy, then we can calculate the time a star can radiate at a given luminosity. This is the Kelvin-Helmholtz timescale and can be shown to be:

$$t_{\rm KH} \approx 30\,{\rm Myr}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-1}\left(\frac{L}{L_\odot}\right)^{-1}. \tag{1.15}$$

- See Problem 1.3
- If a star has no internal energy sources it can generate energy and radiate by contracting. You see this in Computer Problem 1.1.
- In Lord Kelvin's time, this was a problem because we assumed the Sun would be older than this value, since we by then knew the Earth to be at least several billion years old. However, we still didn't know about nuclear energy sources.

1.4.3 Nuclear timescale

- Unit 3 discusses the fusion of hydrogen into helium, where the energy released can be estimated as $\Delta E = \Delta mc^2$. We know about 0.7% of the mass is lost.
- If this fusion process only takes place in the inner 10% of the Sun, the energy available is about $7 \times 10^{-4} Mc^2 = \phi$.
- The rate of change of the energy is again the luminosity $\dot{\phi} = L$.
- The timescale is

$$t_{\rm nuc} = 7 \times 10^{-4}\frac{Mc^2}{L} \tag{1.16}$$

$$= 10^{10}\,{\rm yr}\left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}. \tag{1.17}$$

  We know that luminosity is a strong function of mass, so massive stars burn out their energy very quickly.

1.5 Problems

PROBLEM 1.1: [5 pts]: Verify the statement that where $\varepsilon = 0$ the luminosity is constant using `MESA`, by showing an appropriate/convincing plot from a model.

Answer: The curves in Fig. 1.1 show the luminosity and energy generation rate as functions of interior position for a solar-like model. One sees that at about $0.3\ R_\odot$, the nuclear reactions go to zero, and the luminosity becomes constant.

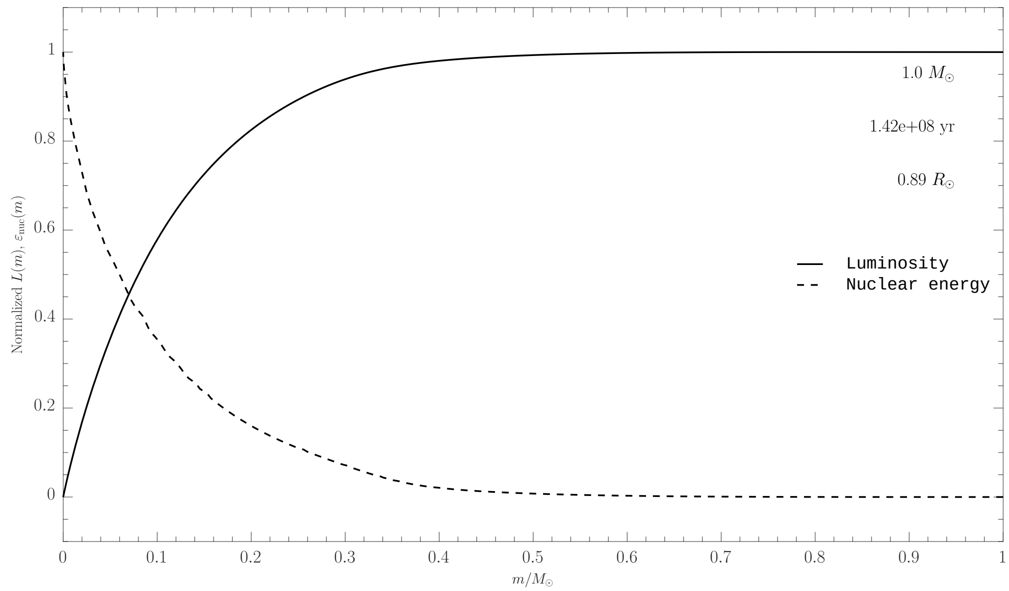

Figure 1.1: Example answer for Problem 1.1. The model profile was actually taken from the one where nuclear reactions were turned on. The luminosity and energy rates are normalized to their largest values. The mass and age are given. Note that this "star" is smaller (in radius) than the current Sun, due to the age.

PROBLEM 1.2: [5 pts]: Derive equation (1.12) by plugging in the constants. Qualitatively, how would the dynamical times scale for a white dwarf star compare to the Sun? A supergiant star?

Answer: A white dwarf star would have a much shorter dynamical timescale since it has a similar mass but a much smaller radius. A supergiant has a much larger radius and likely a large mass but not necessarily that much larger, so it's timescale would be larger than the Sun's.

PROBLEM 1.3: [5 pts]: Derive equation (1.15).

Answer: We know that $L = E/t$ and the gravitational potential energy can be expressed as

$$E = -\frac{GM^2}{R}.$$

Therefore

$$t_{\rm KH} = E/L = \frac{GM^2}{RL} = G\frac{M_\odot^2}{R_\odot L_\odot}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-1}\left(\frac{L}{L_\odot}\right)^{-1}.$$

The coefficient of constants is 9.7×10^{14} s, or about 30 million years (closer to 31).

COMPUTER PROBLEM 1.1: [25 pts]: Here you will look at the effects of "turning off" nuclear reactions at the main sequence to see how stellar evolution changes. `MESA` allows one full control of nuclear energy generation. We will explore how this changes the star right after the time it formed and is getting to the main sequence.
Submit your answers to the questions below with figures. You may also prepare a document with all answers and figures and upload into Canvas.

What to do

1. Copy the `WORK_DIR` to wherever you will be running MESA, rename it to something sensible.

2. Edit the `inlist_project` file. In `&star_job`, make sure `pgstar_flag=.true.` and we don't need to create a pre-main sequence track yet, so you can add `create_pre_main_sequence_model=.false.` In the `&controls` section, we want the `initial_mass=1.0`. Run the model until about 10 billion years, so set `max_age=1d10`. Most importantly, for this first run, turn off the nuclear reaction rates by setting `eps_nuc_factor=0` and `dxdt_nuc_factor=0`.
3. You shouldn't need to run more than about 1000 models (timesteps) to reach that age based on the default dt. All the data gets saved in `LOGS/`.
4. Now copy a new working directory and maybe copy the inlist you just used into it and change the following: Turn on the reactions by setting that variable to 1. We need a stopping criterion, because 10 billion years would take us to the terminal age main sequence, and so we'll use the onset of hyrdrogen burning and set an abundance criterion. So in `&controls` add `xa_central_lower_limit_species(1)='h1'` and then `xa_central_lower_limit(1)=0.69` (the default initial H abundance is 0.7). So right after a little bit of central hydrogen is burned (depleted), the simulation will stop.

Questions

1. How old was the star with nuclear burning when the hydrogen abundance dropped below 0.7? Is that reasonable?

   Answer: The age when the code stops is about 1.42×10^8 years, or 142 million years.
2. Plot a proper HR diagram with the "tracks" of both stars on it (luminosity vs effective temperature, logarithmically). Try to give some indication of age on the plot. You may have to "zoom" in to the appropriate area with a second plot.

   Answer: See Figure 1.2.
3. Describe the two tracks qualitatively.

   Answer: The star with no burning increases in temperature dramatically to about 20000K at relatively fixed luminosity until it reaches 100 million years old or so. Then it cools and gets fainter with time and will continue to do so. The star with burning hardly moves on the HR diagram as it settles into fusion.
4. How long does it roughly take for the model with no nuclear burning to get to its highest T_{eff}. How does that number compare with what is predicted from Equation (1.15). Please show your work.

   Answer: Need to calculate this ...
5. Explain why or how the star with no nuclear burning gets so much hotter than the star with nuclear burning. What is physically happening? Then show a plot that should confirm your explanation that uses interior parameters. What happens for the star with no burning at later times, explain its track on the HR diagram? (Look around for the appropriate quantities to plot in the evolution variables, it's up to you).

   Answer:
   The idea here is that the star with no burning is shrinking because the gravitational pull has no pressure to balance it, and as it shrinks it is heating up. Figure 1.3 shows the central density of both cases as a function of time, confirming the compression. It becomes over 1000 time more dense! We'll see later that the time scale is on the order of the Kelvin-Helmholtz scale. At later times it's just radiating all the heat that built up and cooling. Like a white dwarf at some level.
6. Finally, what causes the star to stop its increase in temperature and head down on the luminosity scale?

   Answer: Degeneracy pressure kicks in, like a white dwarf!
   Some addition plots for other masses and including pre main-sequence modeling are provided in Figs. 1.4 and 1.5.

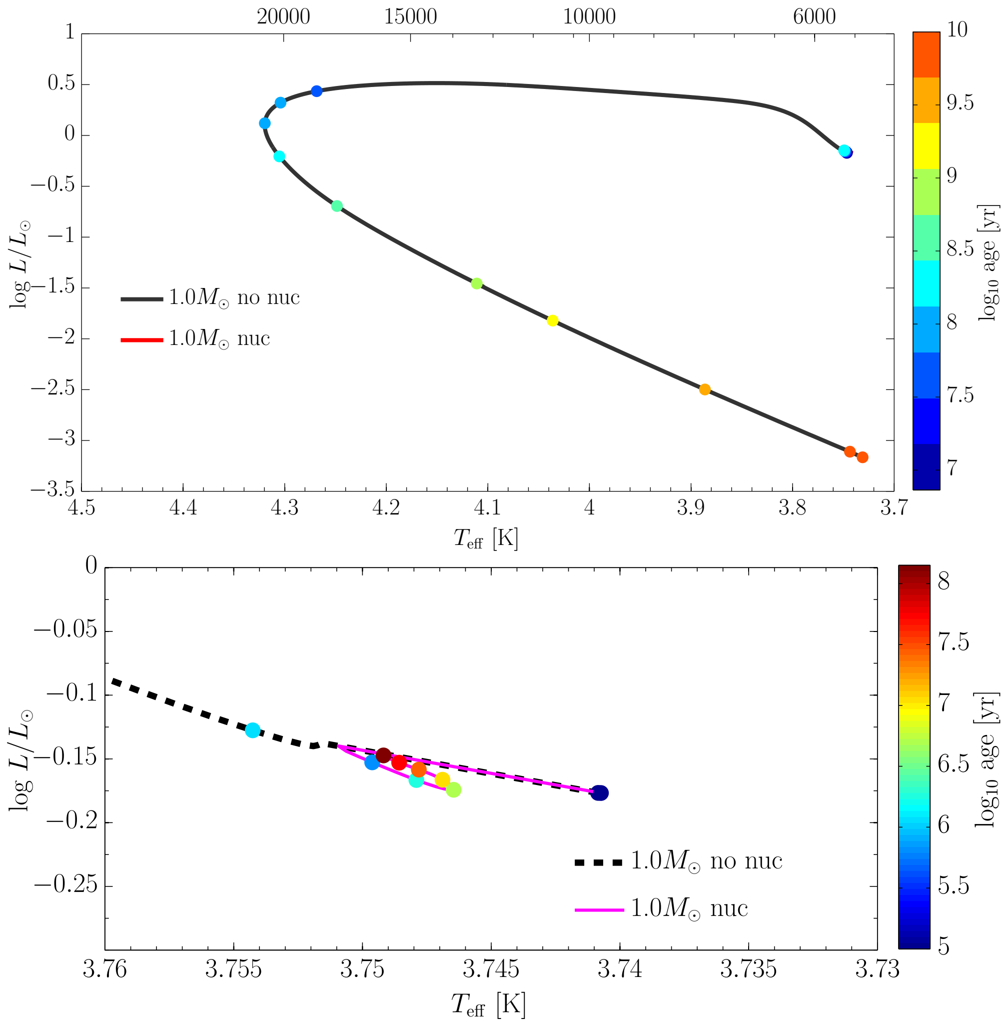

Figure 1.2: HR diagram (top) with the full evolution tracks. The maximum age reaches about 10 billion year. On this scale you cannot discern the star with nuclear reactions. (Bottom) A zoom in near the initial starting point showing the overlap of the stars. The model with nuclear burning does not "move" too much as it settles into burning and reaches hydrostatic equilibrium after only about 100 million years.

What You Should Know How To Do From This Chapter

- Know how to put expressions into convenient forms with the proper coefficient as in the timescale equations.

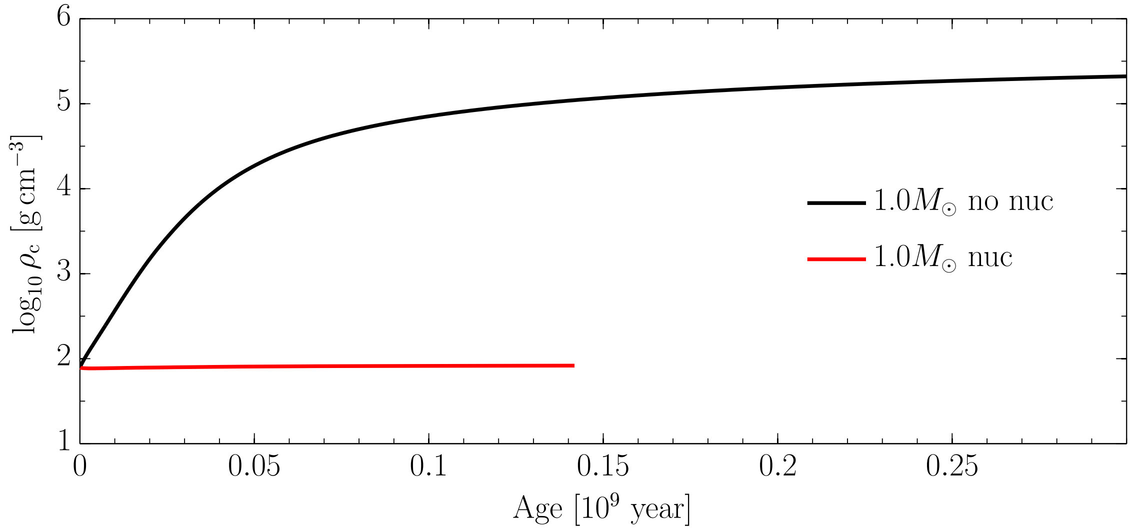

Figure 1.3: Central density ρ_c for each case.

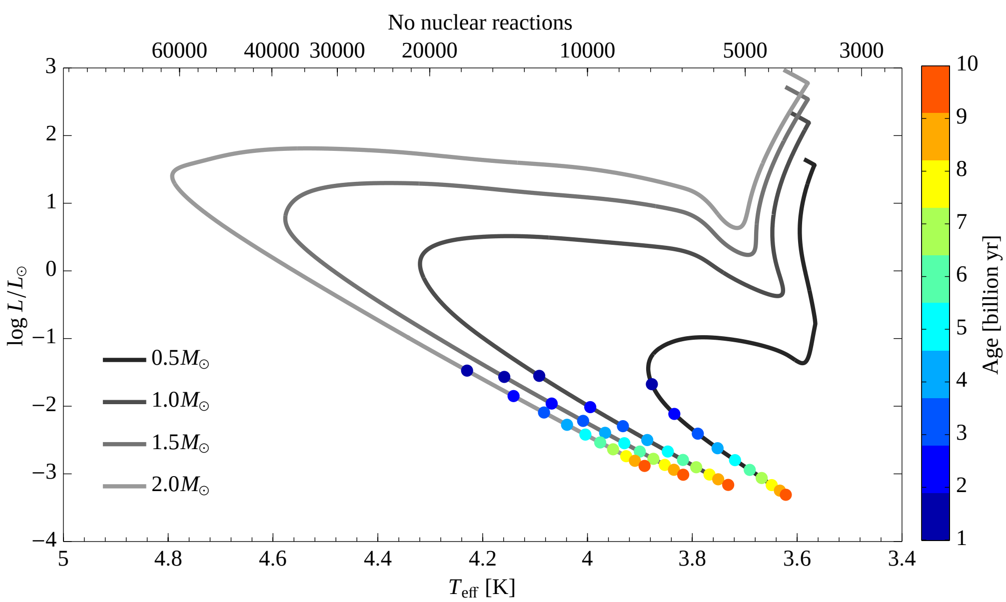

Figure 1.4: Evolutionary tracks of four stars with no nuclear burning and $Z = 0.02$ on an H-R diagram. The stars are evolved for 10 billion years. The ages are given at 1 billion year intervals. The top horizontal axis is the linear temperature scale. Inlists and data are in 2013_05_31_1.

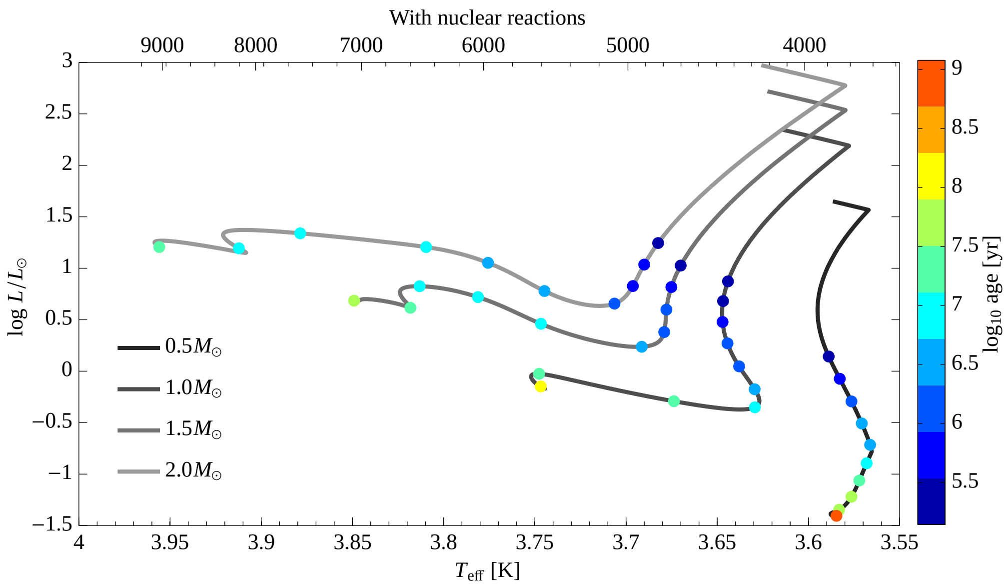

Figure 1.5: Evolutionary tracks of four stars with normal nuclear burning and $Z = 0.02$. The stars are evolved until the central hydrogen fraction is 0.69, from a starting value of 0.7. Note the ages are in $\log_{10}$ scale and there is no unique spacing between the points. Inlists and data are in 2013_05_31_2.

Unit 2

Nuclear interactions

2.1 Coulomb barrier

- So what causes ε to be finite? For stars, it is the *fusion* of 2 nuclei. In order for charged nuclei to fuse, they have to get close together. This is tricky because of the Coulomb barrier

$$E_{\rm C} = \frac{Z_1 Z_2 e^2}{r_0}, \tag{2.1}$$

  where Z_i is the number of protons in the ith nucleus.

- The nucleus has a typical radius of $r_0 \sim 10^{-13}$ cm. The elementary charge $e = 4.8 \times 10^{-10}$ statcoulomb, where a statcoulomb (statC) is $1\,{\rm erg}^{1/2}\,{\rm cm}^{1/2}$. Thus

$$E_{\rm C} = Z_1 Z_2 \frac{(4.8 \times 10^{-10})^2}{10^{-13}} [{\rm erg}] \approx Z_1 Z_2 \cdot 2 \times 10^{-6} [{\rm erg}] \approx Z_1 Z_2 \ [{\rm MeV}]. \tag{2.2}$$

  So for 2 protons ($Z = 1$) the Coulomb barrier is about 1 MeV. 1 erg is about 6.24×10^{11} eV.

- In a star the average kinetic energy per particle (as we'll see) is

$$\langle E_{\rm kin} \rangle = \frac{3}{2} k_{\rm B} T = \frac{3}{2} 1.38 \times 10^{-16}\, T \ [{\rm erg}] \approx 1 \times 10^{-10}\, T \ [{\rm MeV}]. \tag{2.3}$$

  For a temperature of 10 million degrees K, this energy is 2×10^{-9} erg, or about 1 keV. This is 3 orders of magnitude *less* than the amount of energy needed to overcome the Coulomb barrier. That's not looking good.

- What about statistically speaking? The energies of the nuclei are distributed by a Maxwell distribution (in the classical sense): $N(E) \sim \exp(-E/kT)$. For energies 1000 times larger than average energies, their number decreases as $N(E) \sim \exp(-1000)$. There are only about 10^{57} particles in the Sun - the chance of any of having such a large energy is nil.

- According to quantum mechanics, there is a finite probability that a particle can *tunnel* through the Coulomb potential barrier and interact with the other particle. At short distances the nuclear force dominates and is attractive. One solves the Schrödinger Equation. This probability is small, but finite.

2.2 Nuclear reaction rates

- Here, the goal is to derive a general expression for reaction rates between species.

- Let's assume incoming particles can get through the Coulomb barrier at some probability. Understanding the following will help you appreciate stellar modeling codes later on.
- Consider a target particle A and incoming particles a interacting in a box (in the classical sense). Number densities n_a and n_A. Flux of incoming particles is $n_a v$, where v is the relative velocity.
- The number of reactions in the box in time dt are

$$N = \sigma v n_a n_A \mathrm{d}V \mathrm{d}t, \tag{2.4}$$

  where σ is the cross section.
- The cross section is the number of reactions per unit time per target A, divided by the incident flux of particles a. It has units of cm^2.
- Then the *reaction rate* per unit volume is

$$r_{aA} = \sigma v n_a n_A \tag{2.5}$$

- 3 things to worry about. (1) not all incoming particles have the same energy (velocity); (2) the incoming particles have to actually hit the targets, and the cross section can depend very strongly on energy ($\sigma(E)$); (3) an interaction must take place, so the incoming particles have to get near to the targets (get through the barrier). Let's go through these 3 things.

  1. Distribution of energies
     - The v are the relative velocities between incoming and target particles. Assume $f(v)\mathrm{d}v$ denotes the fraction of pairs of particles with speeds between v and $v + \mathrm{d}v$. We should then write

$$r_{aA} = \langle \sigma v \rangle n_a n_A, \tag{2.6}$$

       where

$$\langle \sigma v \rangle = \int_0^\infty \sigma\, v f(v) \mathrm{d}v. \tag{2.7}$$

     - The problem of computing nuclear reaction rates reduces to the evaluation of $\langle \sigma v \rangle$ for the processes that occur in stellar interiors.
     - Let's assume that the distribution is Maxwellian. In velocity space then

$$f(v)\mathrm{d}v = 4\pi \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_\mathrm{B}T}\right) v^2 \mathrm{d}v \tag{2.8}$$

     - Detour: What is m in this equation? Note: Z is the number of protons (atomic number); N is the number of neutrons; A is number of nucleons (protons+neutrons, Z+N, atomic weight). So, for example, the total mass of all incoming particles a is

$$m_a = \sum_i m_{a,i} = Z_a m_\mathrm{p} + N_a m_\mathrm{n} = m_\mathrm{u}(Z_a + (A_a - Z_a)) = A_a m_\mathrm{u}, \tag{2.9}$$

       where m_u is the unified atomic mass unit. Anyway, for particles in relative motion, their kinetic energy is

$$E = \frac{1}{2}\frac{m_a m_A}{m_a + m_A} v^2 = \frac{1}{2}\mathcal{A} m_\mathrm{u} v^2, \tag{2.10}$$

       where that mass combination is the *reduced* mass and $\mathcal{A}$ is the reduced atomic weight

$$\mathcal{A} = \frac{A_a A_A}{A_a + A_A}. \tag{2.11}$$

       The m in Equation (2.8) is therefore $m = \mathcal{A} m_\mathrm{u}$.

– Of course it's more convenient to express Equation (2.8) in terms of energy

$$f(v)\mathrm{d}v = \phi(E)\mathrm{d}E = \frac{2}{\sqrt{\pi}}\frac{E^{1/2}}{(k_\mathrm{B}T)^{3/2}}\exp\left(-\frac{E}{k_\mathrm{B}T}\right)\mathrm{d}E. \tag{2.12}$$

We'll come back to this.

2. So now we have to deal a bit with σ. We first can just consider the geometrical cross section, whose extent is the de Broglie wavelength ($\lambda p = h$)

$$\sigma(E) \propto \pi\lambda^2 \propto p^{-2} \propto E^{-1}. \tag{2.13}$$

3. Finally, the cross section must be a function of the actual probability that the incoming particles indeed tunnel through the Coulomb barrier. Gamow showed that this probability is exponential and proportional to the ratio of Coulomb strength to energy

$$\sigma(E) \propto \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right). \tag{2.14}$$

Notice that only light nuclei will be able to interact at relatively low temperatures (low v).

2.3 Gamow peak

- So putting together these last 2 items we can write an expression for the cross section

$$\sigma(E) \equiv \frac{S(E)}{E}\exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right). \tag{2.15}$$

where $S(E)$ is the cross-section factor, describing the energy dependence of the reaction once the nuclei have penetrated the barrier. Hopefully it varies with energy far less than the rest (which is only true for non-resonant reactions; resonant ones are special cases).

- Using the velocity implied from Equation (2.10) as re-expressed in Equation (2.12), we can show that

$$\sigma(E) \equiv \frac{S(E)}{E}\exp(-bE^{-1/2}), \tag{2.16}$$

where

$$b = 31.291\, Z_1 Z_2\, \mathcal{A}^{1/2}\ \left[\mathrm{keV}^{1/2}\right]. \tag{2.17}$$

- Now Equation (2.7) becomes

$$\langle\sigma v\rangle = \left(\frac{8}{m\pi}\right)^{1/2}(k_\mathrm{B}T)^{-3/2}\int_0^\infty S(E)\mathrm{e}^{-bE^{-1/2}}\mathrm{e}^{-E/k_\mathrm{B}T}\,\mathrm{d}E. \tag{2.18}$$

- Note the two competing effects here: the first exponential increases rapidly with energy, since higher energy nuclei have an easier time to tunnel and this increases the cross section. The second exponential decreases rapidly with energy because of the small probability of there being high energy nuclei. This gives a strongly peaked integrand called the Gamow peak (see sketch).

- The maximum in the curve in energy, known as the "Gamow peak," is then

$$E_0 = \left(\frac{bk_\mathrm{B}T}{2}\right)^{2/3} = 1.22042\,\left(Z_1^2 Z_2^2 \mathcal{A} T_6^2\right)^{1/3}\ [\mathrm{keV}]. \tag{2.19}$$

The notation T_6 is shorthand, in this case, for "millions" of Kelvin. In other words, $T_6 = T \cdot 10^{-6}$, where the real temperature is T.

- Given an $S(E)$, Equation (2.18) can be integrated. A good approximation is to neglect variations of $S(E)$ over the Gamow peak (this doesn't work when there are resonances). We also approximate the rest of the exponential terms by a gaussian function (see Problem 2.3). One finally gets

$$\langle\sigma v\rangle = \frac{8\sqrt{2}}{9\sqrt{3}}\frac{S(E_0)}{\sqrt{mb}}\eta^2 \mathrm{e}^{-\eta}, \tag{2.20}$$

where

$$\eta = \frac{3E_0}{k_\mathrm{B}T} = BT_6^{-1/3}, \tag{2.21}$$

and

$$B = 42.487\left(Z_1^2 Z_2^2 \mathcal{A}\right)^{1/3}. \tag{2.22}$$

- This equation determines the temperature dependence of the average reaction rate. The rates decrease with η, and so increasing $Z_1 Z_2$ decreases the rates, as well as with $\mathcal{A}$, since the velocities decrease at fixed energy. And because η varies as $(Z_1 Z_2)^{2/3}$, the temperature sensitivity of the reactions increases quite strongly with nuclear charge. One thing we've left out is electron screening which increases reaction rates by cancelling some of the positive charge.

- Let's carry on to the actual reaction rates by making a simplification. Let's approximate $\langle\sigma v\rangle$ by its value around some T_0 by

$$\langle\sigma v\rangle \approx \langle\sigma v\rangle_0 \left(\frac{T}{T_0}\right)^n, \tag{2.23}$$

where $\langle\sigma v\rangle_0$ is the value at $T = T_0$.

- The temperature dependence of the cross section is now characterized by

$$n = \frac{\mathrm{d}\ln\langle\sigma v\rangle}{\mathrm{d}\ln T} = \frac{\mathrm{d}\ln\langle\sigma v\rangle}{\mathrm{d}\ln\eta}\frac{\mathrm{d}\ln\eta}{\mathrm{d}\ln T} = \frac{\eta - 2}{3}, \tag{2.24}$$

evaluated at $T = T_0$, which is indeed the temperature dependence of the entire energy release.

- Now we have

$$r_{aA} = \langle\sigma v\rangle_0 n_a n_A \left(\frac{T}{T_0}\right)^n. \tag{2.25}$$

If we re-express the number densities in terms of mass fractions X_i pf the nuclei,

$$n_i = \frac{\rho}{m_\mathrm{u}}\frac{X_i}{A_i}. \tag{2.26}$$

Then

$$r_{aA} = \langle\sigma v\rangle_0 \frac{X_a X_A}{A_a A_A}\frac{\rho^2}{m_\mathrm{u}^2}\left(\frac{T}{T_0}\right)^n. \tag{2.27}$$

We'll come back to this. But note that $\sum_i X_i = 1$.

IN CLASS WORK

Consider 1 hydrogen nucleus and 2 helium nuclei in a $1\,\mathrm{cm}^3$ volume. Using Equation (2.26) compute the number densities of each species, and make sure your answer makes sense.

Answer: The total mass density in this case, with 9 particles, is $\rho = 9\, m_\mathrm{u}\,\mathrm{cm}^{-3}$. We also see that $X_\mathrm{H} = 1/9$ and $X_\mathrm{He} = 8/9$, and $A_\mathrm{H} = 1$ and $A_\mathrm{He} = 4$. . For H, we'd have

$$n_\mathrm{H} = 9\frac{1/9}{1}\,\mathrm{cm}^{-3} = 1\,\mathrm{cm}^{-3}.$$

For He,

$$n_\mathrm{He} = 9\frac{8/9}{4}\,\mathrm{cm}^{-3} = 2\,\mathrm{cm}^{-3}.$$

These particle number densities make sense.

2.4 Problems

PROBLEM 2.1: [10 pts]: First, derive the first equality for E_0 in Equation (2.19). Then show that the constant b in Equation (2.17) is correct. Finally, show that the second equality in Equation (2.19) is correct.

Answer: E_0 is obtained where the first derivative of the exponential term in Equation (2.18) is zero.
To derive the constant b, we plug in the velocity expression to the term in the cross section exponent and find

$$b = \frac{2\sqrt{2}\pi^2 e^2 m_\mathrm{u}^{1/2}}{h} Z_1 Z_2 \mathcal{A}^{1/2},$$

after separating out constants from others. That has units of square root of energy. Using $e^2 = (4.8 \times 10^{-10})^2\,\mathrm{erg\,cm}$, $h = 6.626 \times 10^{-27}\,\mathrm{erg\,s}$, and $m_\mathrm{u} = 931.5 \times 10^3\,\mathrm{keV}/c^2$, the coefficient of 31.3 is found.
The second equality in Equation (2.19) requires a little manipulation. Just counting constants,

$$\left(\frac{(31.291)(8.62 \times 10^{-8}\,\mathrm{keV\,K^{-1}})}{2} T_6 (1 \times 10^6)\right)^{2/3} = 1.22\, T_6^{2/3}.$$

The other non-constant terms follow easily.

PROBLEM 2.2: [5 pts]: Is the energy at the peak of the Gamow curve still consistent with typical nuclei energies in stellar cores? Compute Equation (2.19) for proton-proton collisions at 20 million K, and compare it to the energy of particles in Equation (2.3), both in keV. What does your comparison qualitatively say about the energies of the particles that will participate in reactions with appreciable cross sections?

Answer: E_0 evaluated at this temperature for protons gives about 9 keV (this would increase quite quickly for slightly heavier nuclei). The average kinetic energy per particle using Equation (2.3) is about 2.5 keV, still well below where the peak is. So it's really only the high-energy particles in the tail of the Maxwell distribution that are going to be able to interact with each other.

PROBLEM 2.3: [10 pts]: Show that indeed the exponential terms inside the integral in Equation (2.18) can be approximated as a gaussian function in the vicinity of E_0 when assuming a constant $S(E)$. Thus, show that before integration, the integrand can be expressed as

$$\exp\left(-\frac{3E_0}{k_B T}\right) \exp\left[-\frac{(E-E_0)^2}{2\Delta^2}\right], \tag{2.28}$$

where $\Delta = \sqrt{2E_0 k_\mathrm{B} T/3}$. (Hint: Taylor expand the *argument* of the exponentials around the Gamow peak). You are not being asked to carry out the integration to derive Equation (2.20).

Answer: Consider the argument of the exponential terms in Equation (2.18) as the function

$$f(E) = -\frac{b}{\sqrt{E}} - \frac{E}{k_\mathrm{B} T}.$$

A Taylor expansion around E_0 to second order is generally

$$f(E) \approx f(E_0) + f'(E_0)(E-E_0) + \frac{1}{2} f''(E_0)(E-E_0)^2 + \ldots$$

But note that the second term in the expansion is zero, since that is how we defined E_0 in the first place. So the first term in the expansion is, using the definition for b in Equation (2.19),

$$f(E_0) = -\frac{b}{\sqrt{E_0}} - \frac{E_0}{k_\mathrm{B} T} = -\frac{2E_0}{k_\mathrm{B} T} - \frac{E_0}{k_\mathrm{B} T} = -\frac{3E_0}{k_\mathrm{B} T}.$$

The 2nd-order term, after taking the second derivative and plugging in the b gives

$$\frac{1}{2} f''(E_0)(E-E_0)^2 = -\frac{3}{4 k_\mathrm{B} T E_0}(E-E_0)^2 = -\frac{(E-E_0)^2}{2\Delta^2},$$

which, we put in the exponential, is a classic gaussian function form.

PROBLEM 2.4: [5 pts]: Show that indeed $n = (\eta - 2)/3$ using Equation (2.24).

Answer:

$$\begin{aligned}
\frac{\mathrm{d}\ln\langle\sigma v\rangle}{\mathrm{d}\ln\eta} &= \frac{\eta}{\langle\sigma v\rangle}\frac{\mathrm{d}\langle\sigma v\rangle}{\mathrm{d}\eta} = C\frac{\eta}{\langle\sigma v\rangle}\left(2\eta \mathrm{e}^{-\eta} - \eta^2 \mathrm{e}^{-\eta}\right) = 2 - \eta, \\
\frac{\mathrm{d}\ln\eta}{\mathrm{d}\ln T} &= \frac{T}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}T} = \frac{T}{\eta}\left(-\frac{1}{3}BT^{-4/3}\right) = \frac{T}{\eta}\left(-\frac{1}{3}\frac{\eta}{T}\right) = -\frac{1}{3} \\
\frac{\mathrm{d}\ln\langle\sigma v\rangle}{\mathrm{d}\ln\eta}\frac{\mathrm{d}\ln\eta}{\mathrm{d}\ln T} &= \frac{\eta - 2}{3}
\end{aligned}$$

What You Should Know How To Do From This Chapter

- Know how to Taylor expand a function to a given order so that you end up with an approximate new function that can then be integrated analytically (as in Problem 2.3).

Unit 3

Energy release in nuclear reactions

3.1 Mass excess

- Just recall some definitions (ignoring electrons):
  - Atomic number Z: number of protons in a nucleus. $Z_{\rm H} = 1$, $Z_{\rm He} = 2$, etc. Always an integer.
  - Mass number A: number of protons Z and neutrons N in a nucleus. Always an integer.
  - Atomic mass: The true mass of a single atom (single isotope). A number very nearly equal to the mass number A when expressed in amu. Can be greater or less than A.
  - Atomic weight: The averaged mass over all isotopes of an element (typically what is given in a periodic table). Usually greater than A.
  - Atomic mass unit, amu: 1/12 the mass of neutral carbon 12.
- Consider the reaction of *nuclei*

$$a + A \longrightarrow y + Y. \tag{3.1}$$

  Conservation of energy requires

$$E_{a,A} + (m_a + m_A)c^2 = E_{y,Y} + (m_y + m_Y)c^2, \tag{3.2}$$

  where the $E_{i,I}$ on each side is the kinetic energy of the center-of-mass of each system. Also included are the rest mass energies of each species.

- We can rewrite Equation (3.2) as

$$E_{y,Y} = E_{a,A} + Q, \tag{3.3}$$

  where

$$Q = c^2[m_a + m_A - m_y - m_Y]. \tag{3.4}$$

  The quantity Q can be interpreted as the energy released in any reaction, or, the increase in energy for each reaction.

- Note that these reactions are taking place among *nuclei*. However, since charge is conserved, we may replace the nuclear masses implied in the above equations with the atomic masses, because the same number of electron rest masses will be added to both sides of the equation. A small error in the neglected electron binding energy is introduced (of a few eV), but the great convenience in using atomic masses is well worth it.
- Also conserved in these reactions is the number of *nucleons*, and it is convenient to remove their contribution as it will not change the energy budget. The way to do this is to consider that the mass number is the nearest integer to the exact mass of an atom in atomic mass units.

- So consider an atom with Z protons, N neutrons, and $A = Z + N$ nucleons, with atomic mass m. We can derive what's known as the *mass excess* or *mass defect* Δm (which has units of energy):

$$\begin{aligned}\Delta m &= (m - (Z+N)m_\mathrm{u})c^2, \\ &= (m - Am_\mathrm{u})c^2 \\ &= [m(\mathrm{amu}) - A]\, c^2 m_\mathrm{u},\end{aligned}$$

  or finally,

$$\Delta m = 931.494\,\mathrm{MeV}\,[m(\mathrm{amu}) - A] \quad [\mathrm{MeV}], \tag{3.5}$$

  where $m(\mathrm{amu})$ is the atomic mass of the nucleon in question in amu, and $1 m_\mathrm{u} = 931.5\,\mathrm{MeV/c^2}$.

- Note, mass excess is really just the difference between the atomic mass of an element (which is usually a number with a very small decimal addition) and its mass number (which is always an integer, A).

- Mass excesses are given in the table in Figure 3.6. As an example of using the above expression to come up with these values, take ^{4}He. Its atomic mass $m = 4.002602$. It has 4 nucleons. So

$$\Delta m = (931.494)(4.002602 - 4) = 2.44\,\mathrm{MeV}. \tag{3.6}$$

  (slightly different than the table because of updated atomic masses).

- We can therefore write for the energy release of some reaction in terms of mass excesses:

$$Q_{aA} = [\Delta m(a) + \Delta m(A) - \Delta m(y) - \Delta m(Y)]. \tag{3.7}$$

- In general, the energy is released as kinetic energy to the resultant particles (as implied in Equation (3.3)), and sometimes in photons (and neutrinos, which isn't too important in the energy budget for normal reactions). The energy gets redistributed in the gas through collisions and the absorption of photons. The details don't matter because of energy equilibrium, but what matters is the total amount of heat added to the gas.

- Back to Equation (2.27), the energy generation rate per gram is the reaction rate multiplied by the energy for each reaction divided by density, so

$$\varepsilon_{aA} = \frac{r_{aA} Q_{aA}}{\rho}, \quad [\mathrm{erg\,g^{-1}\,s^{-1}}], \tag{3.8}$$

$$\varepsilon_{aA} = Q_{aA} \langle \sigma v \rangle_0 \frac{X_a X_A}{A_a A_A} \frac{\rho}{m_\mathrm{u}^2} \left(\frac{T}{T_0}\right)^n. \tag{3.9}$$

- One can in general write

$$\varepsilon = \varepsilon_0 \rho T^n, \tag{3.10}$$

  for any reaction, absorbing most of the constant terms in ε_0.

3.2 Binding energy

- Do not confuse mass excess with nuclear binding energy, since they are very similar.

- The nuclear binding energy is the energy required to separate a stable nucleus into its constituent parts, as depicted in Figure 3.1. Note the following definitions:
  - $1\,\mathrm{amu} = 931.494\,\mathrm{MeV}/c^2 = m_\mathrm{u}$
  - $m_\mathrm{p} = 1.007825$ amu
  - $m_\mathrm{n} = 1.00867$ amu

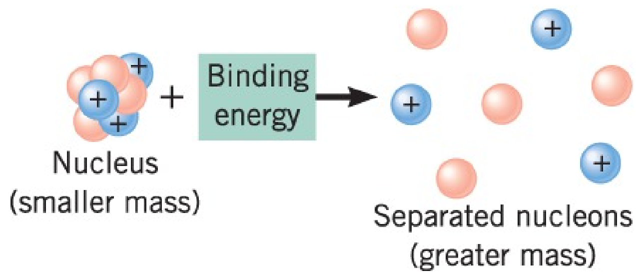

Figure 3.1: Illustration of the concept of binding energy of a nucleus. Typically, a nucleus has a lower energy than if its particles were free. Source of Figure 3.1: http://staff.orecity.k12.or.us/les.sitton/Nuclear/313.htm.

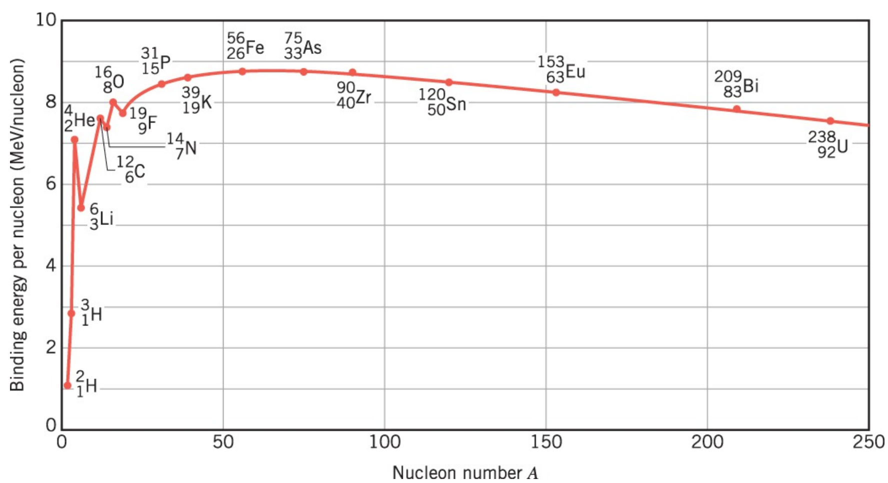

Figure 3.2: The binding energy of isotopes per nucleon. After iron, energy must be used to fuse its nucleons. Source of Figure 3.1: http://staff.orecity.k12.or.us/les.sitton/Nuclear/313.htm.

  – $m_e = 0.0005486$ amu

- Electron binding energy is usually referred to as *ionization energy*.
- Binding energy for neutral atoms is

$$E_{\text{bind}} = 931.494\,\text{MeV}/\text{c}^2\,(Zm_p + Nm_n + Zm_e - m)\,c^2 \qquad [\text{MeV}], \tag{3.11}$$

  where m is the atomic mass of the element or isotope in question. Often you'll see the binding energy per nucleon, E_{bind}/A. Electrons can be ignored when talking about nuclei. This is plotted in Fig. 3.2.

- The atomic mass of ^{4}He is 4.0026 amu. It's binding energy per nucleon is $931.494 * (2m_p + 2m_n - 4.0026)/4 = 7.07$ MeV/nucleon.
- The large binding energy of ^{4}He is important in stellar physics, as we'll see.

3.3 Hydrogen burning

- Historically, considering a star of H and He, all major 2-particle interactions produce unstable nuclei

$$\begin{aligned} \text{p} + \text{p} &\longrightarrow {}^2\text{He (unstable)} \longrightarrow \text{p} + \text{p} \\ \text{p} +^4\text{He} &\longrightarrow {}^5\text{Li (unstable)} \longrightarrow \text{p} +^4\text{He} \\ {}^4\text{He} +^4\text{He} &\longrightarrow {}^8\text{Be (unstable)} \longrightarrow {}^4\text{He} +^4\text{He}. \end{aligned} \tag{3.12}$$

- It was Hans Bethe who first showed that the weak force plays a role in all this, in the form of beta decay (see below).
- Anyway, the general idea of hydrogen fusion is always

$$4^1\mathrm{H} \longrightarrow^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e}, \tag{3.13}$$

where 2 positrons are needed to keep charge conserved, and 2 electron neutrinos conserve lepton number (from the 2 anti-lepton positrons). This does not happen all at once, but along certain "paths," see below.

- The atomic mass of H is 1.007852 amu and of ^{4}He is 4.002603 amu. So 0.0288 mass units are lost. The mass fraction that is turned into energy is thus $0.0288/4 = 0.007$, or 0.7%.
- Using Δmc^2, we have $(0.0288)(931.494\,\mathrm{MeV}/c^2)c^2 = 26.8\,\mathrm{MeV}$.
- Using the mass excesses in the units of MeV we can compute the energy liberated in yet another way

$$Q = 4(7.289) - 2.4248 - 2(0.263) = 26.21\,\mathrm{MeV}, \tag{3.14}$$

where this time we take into account the energy carried away by the neutrinos (see table in Fig. 3.3). Note, as mentioned before, that the values are from atomic mass excesses, not *nuclear* mass excesses, so electrons are implicitly in there (including . That is why we do not take into account the ~ 0.5 MeV from the positrons (since those, plus the 2 electrons on the RHS implicit in the the He, cancel energetically with the 4 electrons implicit on the LHS).

- Let's be redundant, and clear. Consider the energy available to the star (sometimes referred to as the effective energy) for a generic β decay process, whereby a proton converts to a neutron in some nucleus, producing a positron and neutrino as well. This energy can be written as (using our notation)

$$\begin{aligned} Q &= \Delta m_\mathrm{nuc}(Z+1) - \Delta m_\mathrm{nuc}(Z) - m_e c^2 + 2m_e c^2 - E_\nu, \\ &= \Delta m_\mathrm{nuc}(Z+1) - \Delta m_\mathrm{nuc}(Z) + m_e c^2 - E_\nu, \\ &= \Delta m_\mathrm{atom}(Z+1) - \Delta m_\mathrm{atom}(Z) - E_\nu, \end{aligned}$$

where the Δm are the mass excesses (in energy units), Z is the proton number of the nucleus, E_ν is the energy carried off by the neutrino (not available to the star), the $-m_e c^2$ is the energy required to create the positron, and the $+2m_e c^2$ is the energy produced when the electron-positron annihiliation takes place (gamma radiation is typically produced). At one point we switched from using "nuclear" mass excesses to "atomic" ones (as the ones in the tables) which take into account electron contributions. That's why the electrons are "already counted" in the energy budget, at some level.

3.3.1 PP-I chain

- The proton-proton reaction is

$$\begin{aligned} ^1\mathrm{H} + ^1\mathrm{H} &\longrightarrow {}^2\mathrm{D} + \mathrm{e}^+ + \nu = 2(7.289) - 13.136 - 0.263 = 1.18\,\mathrm{MeV};\ (10^9\,\mathrm{yr}) \\ ^2\mathrm{D} + ^1\mathrm{H} &\longrightarrow {}^3\mathrm{He} + \gamma = 13.136 + 7.289 - 14.93 = 5.49\,\mathrm{MeV};\ (6\,\mathrm{s}), \\ ^3\mathrm{He} + ^3\mathrm{He} &\longrightarrow {}^4\mathrm{He} + ^1\mathrm{H} + ^1\mathrm{H} = 2(14.93) - 2.42 - 2(7.289) = 12.86\,\mathrm{MeV};\ (10^6\,\mathrm{yr}) \end{aligned}$$

Noting that the first two reactions have to happen twice to produce 2 ^{3}He nuclei, the total energy is $2(1.18)+2(5.49)+12.86 = 26.2$ MeV, as we saw before.

- Another way to write this is

$$^1\mathrm{H}(^1\mathrm{H}, \mathrm{e}^+\nu_\mathrm{e})^2\mathrm{D}(^1\mathrm{H}, \gamma)^3\mathrm{He}(^3\mathrm{He}, 2^1\mathrm{H})^4\mathrm{He}, \tag{3.15}$$

where everything to the left of the comma is an ingredient of the reaction, and everything to the right is a product.

Table 5-1 Reactions of the PP chains

Reaction	Q value, Mev	Average ν loss, Mev	S_0, kev barns	$\frac{dS}{dE}$, barns	B	τ_{12}, years†
$H^1(p,\beta^+\nu)D^2$	1.442	0.263	3.78×10^{-22}	4.2×10^{-24}	33.81	7.9×10^{9}
$D^2(p,\gamma)He^3$	5.493		2.5×10^{-4}	7.9×10^{-6}	37.21	4.4×10^{-8}
$He^3(He^3,2p)He^4$	12.859		5.0×10^{3}		122.77	2.4×10^{5}
$He^3(\alpha,\gamma)Be^7$	1.586		4.7×10^{-1}	-2.8×10^{-4}	122.28	9.7×10^{5}
$Be^7(e^-,\nu)Li^7$	0.861	0.80				3.9×10^{-1}
$Li^7(p,\alpha)He^4$	17.347		1.2×10^{2}		84.73	1.8×10^{-5}
$Be^7(p,\gamma)B^8$	0.135		4.0×10^{-2}		102.65	6.6×10^{1}
$B^8(\beta^+\nu)Be^{8*}(\alpha)He^4$	18.074	7.2				3×10^{-8}

† Computed for $X = Y = 0.5$, $\rho = 100$, $T_6 = 15$ (sun).

Figure 3.3: The properties of the relevant chains of the proton-proton reaction. From Clayton [1983].

- See the table in Figure 3.3 for quantities, particularly the neutrino loss contribution. The times given apply to a single nucleus in the stellar interior environment.
- Neutrino production is the main reason why we know these processes are taking place in stellar interiors.
- The onset of H burning through this channel can occur at about 5-10 MK.
- The first part of the chain is by far the slowest, because a proton is converting into a neutron through the weak force (beta decay), and this is quite rare.
- Note this chain can occur in a pure hydrogen gas.
- The first reaction in the chain goes the slowest and so the rate of energy generation is controlled by it.
- The deuterium burning reaction is very fast, and stars should destroy all of it. The large abundance on Earth is an interesting problem.

3.3.2 PP-II and PP-III chains

- After the second step of the PP-I chain, the ^{3}He that was produced has a choice, which is dictated mainly by temperature.
- Helium 4 can also be produced from hydrogen in 2 other ways.
- PP-II chain:
$$^3\mathrm{He}(^4\mathrm{He},\gamma)^7\mathrm{Be}(\mathrm{e}^-,\bar{\nu}_\mathrm{e})^7\mathrm{Li}(^1\mathrm{H},^4\mathrm{He})^4\mathrm{He} \tag{3.16}$$
- PP-III chain:
$$^7\mathrm{Be}(^1\mathrm{H},\gamma)^8\mathrm{B}(,\mathrm{e}^+\nu_\mathrm{e})^8\mathrm{Be}(,^4\mathrm{He})^4\mathrm{He} \tag{3.17}$$
- Note that the berrylium 7 nucleus has a choice to react with an electron (to form lithium 7) or a proton (to form berrylium 8).
- For temperatures above about 15 million K, helium 3 likes to react with helium 4 (rather than itself as in the last reaction of PP-I) and so the PP-I chain is not as dominant at hotter conditions.
- The average neutrino energy loss from PP-II in the berrylium electron capture is 0.8 MeV.
- The average neutrino energy loss from PP-III in the positron decay of boron is 7.2 MeV, which is large.

Table 5-2 The CNO reactions

Reaction	Q value, Mev	Average ν loss, Mev	$S(E=0)$, kev barns	$\frac{dS}{dE}$, barns	B
$C^{12}(p,\gamma)N^{13}$	1.944		1.40	4.26×10^{-3}	136.93
$N^{13}(\beta^+\nu)C^{13}$	2.221	0.710			
$C^{13}(p,\gamma)N^{14}$	7.550		5.50	1.34×10^{-2}	137.20
$N^{14}(p,\gamma)O^{15}$	7.293		2.75		152.31
$O^{15}(\beta^+,\nu)N^{15}$	2.761	1.00			
$N^{15}(p,\alpha)C^{12}$	4.965		5.34×10^{4}	8.22×10^{2}	152.54
$N^{15}(p,\gamma)O^{16}$	12.126		2.74×10^{1}	1.86×10^{-1}	152.54
$O^{16}(p,\gamma)F^{17}$	0.601		1.03×10^{1}	-2.81×10^{-2}	166.96
$F^{17}(\beta^+\nu)O^{17}$	2.762	0.94			
$O^{17}(p,\alpha)N^{14}$	1.193		Resonant reaction		167.15

Figure 3.4: The properties of the relevant chains of the CNO cycle. From Clayton [1983].

- These neutrinos coming from the Sun can be detected and have been critical to understanding fusion processes. They are the dominant ones observed in the water tank experiments and the cleaning fluid experiment: $\nu_e(^{37}\mathrm{Cl}, e^-)^{37}\mathrm{Ar}$.

- The branching ratio among these three chains depends quite sensitively on interior conditions.

3.3.3 CNO cycle

- Another way to turn hydrogen into helium is by producing (and burning) carbon, nitrogren, and oxygen, as long as such species exist. This typically happens in higher-mass stars with higher internal temperatures.

- There are two possible ways this happens, each involving 6 reactions. This bi-cycle is written as

$$^{12}\mathrm{C}(^1\mathrm{H},\gamma)^{13}\mathrm{N}(,e^+\nu_e)^{13}\mathrm{C}(^1\mathrm{H},\gamma)^{14}\mathrm{N}(^1\mathrm{H},\gamma)^{15}\mathrm{O}(,e^+\nu_e)^{15}\mathrm{N}(^1\mathrm{H},^4\mathrm{He})^{12}\mathrm{C}, \tag{3.18}$$

$$^{14}\mathrm{N}(^1\mathrm{H},\gamma)^{15}\mathrm{O}(,e^+\nu_e)^{15}\mathrm{N}(^1\mathrm{H},\gamma)^{16}\mathrm{O}(^1\mathrm{H},\gamma)^{17}\mathrm{F}(,e^+\nu_e)^{17}\mathrm{O}(^1\mathrm{H},^4\mathrm{He})^{14}\mathrm{N}. \tag{3.19}$$

- The nucleon making 2 choices is ^{15}N upon interaction with a proton.

- The energy released is $Q = 25.02$ MeV.

- For Pop. I stars, CNO material makes up about 75% of metals, and O is about 70% of CNO.

- The slowest reaction and the one that determines the overall reaction rate is the proton capture $^{14}\mathrm{N}(^1\mathrm{H},\gamma)^{15}\mathrm{O}$. This reaction is highly temperature dependent (Problem 3.4).

- See Figure 3.5 for the crossover regime between PP and CNO channels.

- As this is the slowest reaction, the most abundant species in the CNO cycle is ^{14}N.

- Generally, through these channels, the final abundances of C and O decrease, while that of N increases.

- We'll come to He burning later in the post main-sequence unit

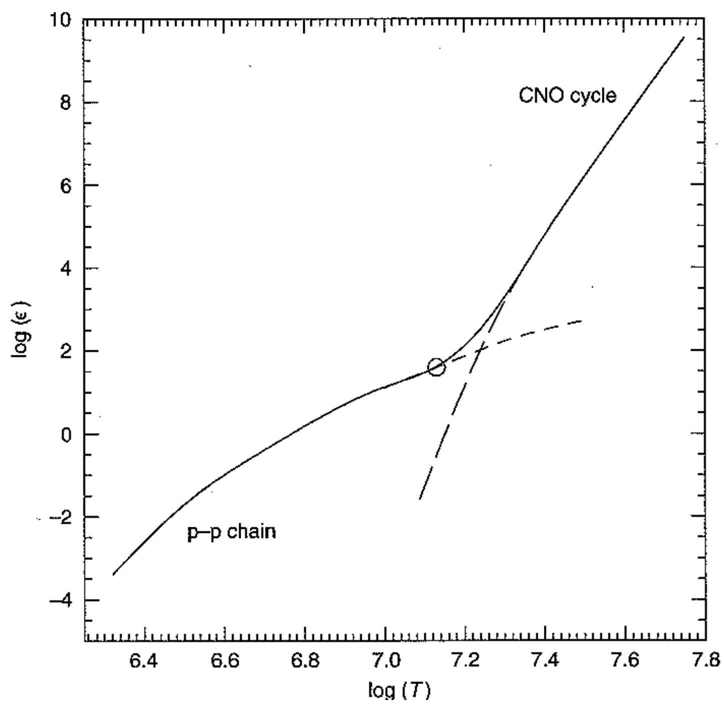

Figure 3.5: The nuclear energy rate as a function of temperature. The Sun is marked as a circle. From Salaris and Cassisi [2006].

3.4 Problems

PROBLEM 3.1: [5 pts]: The Sun's luminosity is $L_\odot = 3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}}$. Assume that the energy for this luminosity is provided solely by the PP-I chain, and that neutrinos carry off 3% of the energy liberated. How many neutrinos are produced per second? What is the neutrino flux at Earth (# neutrinos per second per cm^2)?

Answer: The number of reactions per second is the luminosity (energy per second) divided by the amount of energy in 1 reaction. The total energy in PP-I (ignoring the neutrinos) is 26.731 MeV. If neutrinos take away 3% of that, we find

$$N_\nu = \frac{3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}} \times 6.24 \times 10^{11}\,\mathrm{eV\,erg^{-1}}}{(26.73 - 0.03 * 26.73) \times 10^6\,\mathrm{eV}} = 9.4 \times 10^{37}\,\nu\,\mathrm{sec}^{-1}.$$

The flux at $1\,\mathrm{AU} = 1.5 \times 10^{13}\,\mathrm{cm}$ is $N_\nu/4\pi r^2 = 3.3 \times 10^{10}\,\nu\,\mathrm{s^{-1}\,cm^{-2}}$. There might be a factor of 2 missing here due to 2 neutrinos being created.

PROBLEM 3.2: [5 pts]: Show that at a temperature of $T = 15$ MK the temperature exponent (Equation (2.24)) in the first reaction of the PP-I chain is $n \simeq 4$.

Answer: To compute n, we need η, and thus B and A, the reduced atomic weight. The first reaction in PP-I is just 2 protons so $Z_1 = Z_2 = 1$, and $A_1 = A_2 = 1$, so $A = 1/2$. With the given temperature, this comes out to $n \simeq 3.9$.

PROBLEM 3.3: [10 pts]: Compute the *full* Q values for the PP-II and PP-III chains (don't forget to include any necessary contributions from previous chains in the computation).

Answer: We can use the table of mass excesses, or, more easily, the table in Fig. 3.3.
For PP-II, we need to remember that the first 2 reactions in the PP-I chain are necessary so that there is helium 3 available. Thus there is already $1.18 + 5.49 = 6.67\,\mathrm{MeV}$ available. Then

$$\begin{aligned} {}^{3}\mathrm{He} + {}^{4}\mathrm{He} &\longrightarrow {}^{7}\mathrm{Be} + \gamma = 1.586\,\mathrm{MeV}; \\ {}^{7}\mathrm{Be} + \mathrm{e}^{-} &\longrightarrow {}^{7}\mathrm{Li} + \nu_{\mathrm{e}} = 0.861 - 0.8 = 0.061\,\mathrm{MeV}; \\ {}^{7}\mathrm{Li} + {}^{1}\mathrm{H} &\longrightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He} = 17.347\,\mathrm{MeV}. \end{aligned}$$

The total is thus $Q_{\mathrm{pp-II}} = 25.66\,\mathrm{MeV}$.
For PP-III, the same holds true for the first 2 reactions of PP-I. In addition, the first reaction of PP-II needs to be included so that berrylium 7 is presnt. The total of these 3 is $6.67 + 1.59 = 8.26\,\mathrm{MeV}$. We'll do this one using the mass excess table. Then we have

$$\begin{aligned} {}^{7}\mathrm{Be} + {}^{1}\mathrm{H} &\longrightarrow {}^{8}\mathrm{B} + \gamma = 15.769 + 7.29 - 22.923 = 0.136\,\mathrm{MeV}; \\ {}^{8}\mathrm{B} &\longrightarrow {}^{8}\mathrm{Be} + \mathrm{e}^{+} + \nu_{\mathrm{e}} = 22.923 - 4.94 - 7.2 = 10.783\,\mathrm{MeV}; \\ {}^{8}\mathrm{Be} &\longrightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He} = 4.94 - 2(2.42) = 0.1\,\mathrm{MeV}. \end{aligned}$$

The total is $Q_{\mathrm{pp-III}} = 19.28\,\mathrm{MeV}$.

PROBLEM 3.4: [5 pts]: Show that at a temperature of $T = 15$ MK the temperature exponent in the slowest reaction of the CNO cycle is $n \simeq 20$.

Answer: We need the same quantities as discussed in Prob. 3.2. The slowest reaction involves nitrogen 14 and a proton. So $Z_1 = 7$, $Z_2 = 1$, and $A_1 = 14$, $A_2 = 1$. With the given temperature, this comes out to $n \simeq 19.9$.

What You Should Know How To Do From This Chapter

- Understand how to compute the temperature dependence of nuclear reactions (given the specific formula)

Table 4-1 Atomic mass excesses†

Z	*Element*	A	$M - A$, *Mev*
0	n	1	8.07144
1	H	1	7.28899
	D	2	13.13591
	T	3	14.94995
	H	4	28.22000
		5	31.09000
2	He	3	14.93134
		4	2.42475
		5	11.45400
		6	17.59820
		7	26.03000
		8	32.00000
3	Li	5	11.67900
		6	14.08840
		7	14.90730
		8	20.94620
		9	24.96500
4	Be	6	18.37560
		7	15.76890
		8	4.94420
		9	11.35050
		10	12.60700
		11	20.18100
5	B	7	27.99000
		8	22.92310
		9	12.41860
		10	12.05220
		11	8.66768
		12	13.37020
		13	16.56160
6	C	9	28.99000
		10	15.65800
		11	10.64840
		12	0
		13	3.12460
		14	3.01982
		15	9.87320
7	N	12	17.36400
		13	5.34520
		14	2.86373
		15	0.10040
		16	5.68510
		17	7.87100
8	O	14	8.00800
		15	2.85990
		16	−4.73655
		17	−0.80770
		18	−0.78243
		19	3.33270
		20	3.79900
9	F	16	10.90400
		17	1.95190
		18	0.87240
		19	−1.48600
		20	−0.01190
		21	−0.04600
10	Ne	18	5.31930
		19	1.75200
		20	−7.04150
		21	−5.72990
		22	−8.02490
		23	−5.14830
		24	−5.94900
11	Na	20	8.28000
		21	−2.18500
		22	−5.18220
		23	−9.52830
		24	−8.41840
		25	−9.35600
		26	−7.69000
12	Mg	22	−0.14000
		23	−5.47240
		24	−13.93330
		25	−13.19070
		26	−16.21420
		27	−14.58260
		28	−15.02000
13	Al	24	0.1000
		25	−8.9310
		26	−12.2108
		27	−17.1961
		28	−16.8554
		29	−18.2180
		30	−17.1500
14	Si	26	−7.1320
		27	−12.3860
		28	−21.4899
		29	−21.8936
		30	−24.4394
		31	−22.9620
		32	−24.2000
15	P	28	−7.6600
		29	−16.9450
		30	−20.1970
		31	−24.4376
		32	−24.3027

Figure 3.6: Mass excesses from Clayton [1983].

Unit 4

Distribution functions

In Unit 6 we will start to derive equations of state of stellar matter. To do so from first principles, we need to know how, in general, particles are distributed as a function of momentum (or energy). This requires some basic statistical mechanics.

Statistical mechanics deals with the occupation of energy states when a system is excited. The fundamental assumption of statistical mechanics is that, in thermal equilibium, every distinct state with the same total energy is occupied with equal probability. Temperature is simply a measure of the total energy of a system in thermal equilibium. The only change from classical statistical mechanics to quantum mechanics has to do with how we count distinct states, which depends on whether the particles involved are distinguishable, identical fermions, or identical bosons.

4.1 An example

- Consider 3 non-interacting particles of equal mass in some potential.
- The total energy of the system is 243 (with some arbitrary energy units). This means that the particles occupy some energy levels n_i such that, using a simple and arbitrary energy rule, we have

$$E_{\text{total}} = \sum_{i=1}^{\infty} n_i i^2 = 243. \tag{4.1}$$

  A one-dimensional square well, for example, has a dispersion relation like this one. There would be prefactors before the summation symbol to give the right units of energy; for simplicity, ignore such terms.

- Consider distinguishable (classical) particles. There are 13 unique ways of distributing 3 particles into various energy levels to get a total energy of 243.
  - We can have all 3 particles in the 9th state: $n_9 = 3$. There is only 1 way of doing this.
  - $n_1 = 1,\ n_{11} = 2.$ (3 ways)
  - $n_3 = 2,\ n_{15} = 1.$ (3 ways)
  - $n_5 = n_7 = n_{13} = 1.$ (6 ways)
- Quantum statistics says, for large N, that all states with the same N and same E_{total} are equally likely.
- Therefore, in thermal equilibrium, the most probable configuration is the one that can be achieved in the largest number of ways. The last state is this case. We'll come back to this.
- Now consider fermions, which are indistinguishable, and cannot occupy the same state.

- There is only one possibility here: $n_5 = n_7 = n_{13} = 1$.
- Finally, consider bosons, which are indistinguishable. There are 3 distinct states:
  - $n_9 = 3$. (1 way)
  - $n_3 = 2,\ n_{15} = 1$. (1 way)
  - $n_5 = n_7 = n_{13} = 1$. (1 way)

4.2 Partition function

- Again consider N particles with the same masses. There are energy states E_i with degeneracies g_i (distinct states with same energy E_i). We distribute the N particles such that there are N_1 particles with energy E_1, N_2 particles with energy E_2, etc. We want to know how many different ways we can do this?
- We now define $Q(n_1, n_2, n_3, \ldots)$ to be the number of microscopically distinguishable arrangements that lead to the same macroscopic distribution.
- It is sometimes known as a partition function, or probability distribution function, or canonical ensemble, etc.
- Clearly, Q depends strongly on the type of particle we are considering, as the 3 cases below show (without derivation).

4.2.1 Distinguishable particles

In this case,

$$Q = N! \prod_{i=1}^{\infty} \frac{g_i^{n_i}}{n_i!}$$

To prove this, we go back to the previous illustration:

$$Q(n_9 = 3) = 3! \left(\frac{1^3}{3!}\right) = 1. \tag{4.2}$$

$$Q(n_3 = 2, n_{15} = 1) = 3! \left(\frac{1^2}{2!}\right)\left(\frac{1^1}{1!}\right) = 3. \tag{4.3}$$

$$Q(n_5 = 1, n_7 = 1, n_{13} = 1) = 3! \left(\frac{1^1}{1!}\right)\left(\frac{1^1}{1!}\right)\left(\frac{1^1}{1!}\right) = 6. \tag{4.4}$$

This confirms our earlier counting.

4.2.2 Identical fermions

In this case,

$$Q = \prod_{i=1}^{\infty} \frac{g_i!}{n_i!\,(g_i - n_i)!}.$$

So,

$$Q(n_5 = 1, n_7 = 1, n_{13} = 1) = \left(\frac{1!}{1!0!}\right)\left(\frac{1!}{1!0!}\right)\left(\frac{1!}{1!0!}\right) = 1. \tag{4.5}$$

4.2.3 Identical bosons

In this case,

$$Q = \prod_{i=1}^{\infty} \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}.$$

4.3 Derivation

In thermal equilibirum, each energy state with some occupying number of particles is equally likely. The most probable configuration is one that can be obtained with the largest number of different ways, such that $Q(n_i)$ is **maximum**. The only constraints in this problem are that the total number of particles in each state add up to the total, or

$$\sum_i n_i = N,$$

and that the total energy is maintained as

$$\sum_i n_i E_i = E_{\text{tot}}.$$

To solve such a problem we can introduce a new function and Lagrange multipliers that help maintain the constraints. Instead of maximizing Q, it's more convenient to consider $\ln Q$. So we want to maximize

$$G = \ln Q + \alpha \left[N - \sum_i n_i \right] + \beta \left[E_{\text{tot}} - \sum_i n_i E_i \right].$$

Then to find the maximum we compute $\partial G / \partial n_j = 0$. Also, regarding the Lagrange multipliers, $\partial G / \partial \alpha = 0$ and $\partial G / \partial \beta = 0$ simply reproduce the constraints.

The quantity Q needs to be considered for the 3 different types of particles that we've discussed.

It will also be helpful to utilize Stirling's approximation:

$$\ln(x!) \approx x \ln x - x, \tag{4.6}$$

which holds when $x \gg 1$.

1. Distinguishable particles. Do this in detail. Using our Q in this case we have

$$\begin{aligned} G &= \ln N! + \ln \prod_{i=1} \frac{g_i^{n_i}}{n_i!} + \alpha \left[N - \sum_i n_i \right] + \beta \left[E - \sum_i n_i E_i \right] \\ &= \ln N! + \sum_i \ln \frac{g_i^{n_i}}{n_i!} + \alpha \left[N - \sum_i n_i \right] + \beta \left[E - \sum_i n_i E_i \right] \\ &= \ln N! + \sum_i n_i \ln g_i - \sum_i \ln n! + \alpha \left[N - \sum_i n_i \right] + \beta \left[E - \sum_i n_i E_i \right] \\ &= N \ln N - N + \sum_i n_i \ln g_i - \sum_i n_i \ln n_i + \sum_i n_i + \alpha \left[N - \sum_i n_i \right] + \beta \left[E - \sum_i n_i E_i \right]. \end{aligned}$$

   Note $E \equiv E_{\text{tot}}$.

Now taking the partial derivative

$$\begin{aligned}\frac{\partial G}{\partial n_j} &= \ln g_j - \ln n_j - \frac{n_j}{n_j} + 1 - \alpha - \beta E_j = 0 \\ &= \ln g_j - \ln n_j - \alpha - \beta E_j = 0 \\ \ln \frac{g_j}{n_j} &= \alpha + \beta E_j \\ \frac{g_j}{n_j} &= e^{\alpha + \beta E_j} \\ n_j &= g_j e^{-\alpha - \beta E_j}.\end{aligned}$$

This is the result we are looking for.

2. Fermions. Following the same procedure, we find

$$n_j = \frac{g_j}{\exp(\alpha + \beta E_j) + 1}.$$

3. Bosons. Again, the same procedure yields

$$n_j = \frac{g_j}{\exp(\alpha + \beta E_j) - 1}.$$

To determine what α and β are, one needs to plug the n_j into the constraints and consider some specific total particle number N and energy system E. One then finds that

$$\begin{aligned}\alpha &= -\frac{\mu}{k_{\mathrm{B}}T}, \\ \beta &= \frac{1}{k_{\mathrm{B}}T},\end{aligned}$$

where μ is the chemical potential and k_{B} is Boltzmann's constant.

Unit 5

Mean molecular weight

- Another concept before we dive into equations of state is the mean molecular weight μ, which is important to understand.
- Stellar interiors have a mixture of atoms of different elements and various ionizations.
- Consider the mean mass $\overline{m}$ per particle

$$\overline{m} = \frac{\sum_j n_{j,\mathrm{I}} m_{j,\mathrm{I}} + n_\mathrm{e} m_\mathrm{e}}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}} \approx \frac{\sum_j n_{j,\mathrm{I}} m_{j,\mathrm{I}}}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}}, \tag{5.1}$$

where $n_{j,\mathrm{I}}$ is the ion number density of ion j, $m_{j,\mathrm{I}}$ is its mass, and n_e and m_e are the numbers and mass of the electron (and then we ignore the electron mass).

- The mass of the jth ion is approximately its number of protons and neutrons (A_j) times the amu, or $m_{j,\mathrm{I}} = A_j m_\mathrm{u}$.
- So then we define

$$\mu = \frac{\overline{m}}{m_\mathrm{u}} = \frac{\sum_j n_{j,\mathrm{I}} A_j}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}}. \tag{5.2}$$

This can be interpreted as the average mass per particle (ion, electron, etc.) in units of the amu.

- Note that the total particle number density in the gas is

$$n = n_e + n_\mathrm{I} = n_e + \sum_j n_{j,\mathrm{I}} = \sum_j (1 + Z_j) n_{j,\mathrm{I}}, \tag{5.3}$$

since one ionized atom contributes 1 nucleus plus Z_j electrons. The total $n_\mathrm{e} = \sum_j n_{j,\mathrm{I}} Z_j$, where Z_j is the "charge" of each nucleus.

- In general though, the electron density (or level of ionization) is complicated and derived from the Saha equation. Such an equation gives ionization fractions y_i such that the electron number density would be $n_\mathrm{e} = \sum_j n_{j,\mathrm{I}} y_j Z_j$ (see later).
- But to be more useful, it's easier to express the number densities in terms of mass fractions x_i, where $\sum_i x_i = 1$.
- The number densities we looked at earlier are for some species i are

$$n_i = \frac{\rho}{m_\mathrm{u}} \frac{x_i}{A_i}. \tag{5.4}$$

Think of this as the mass per unit volume of species i (ρx_i), in units of 1 ion of species i ($m_\mathrm{u} A_i$).

IN CLASS WORK

Imagine a star where 92% of all particles are hydrogen nuclei and 8% of them are helium nuclei. What are the mass fractions of hydrogen and helium?

$$\begin{aligned}
92 &= \frac{\rho x_{\rm H}}{m_{\rm u}} \\
8 &= \frac{\rho x_{\rm He}}{4m_{\rm u}} \\
x_{\rm H} &= 92m_{\rm u}/\rho \\
x_{\rm He} &= 32m_{\rm u}/\rho \\
92m_{\rm u}/\rho + 32m_{\rm u}/\rho &= 124m_{\rm u}/\rho = 1 \\
m_{\rm u}/\rho &= 1/124 \\
x_{\rm H} &= 92/124 = 0.7419 \\
x_{\rm He} &= 32/124 = 0.2581.
\end{aligned}$$

- So using this, we now have

$$\mu = \frac{\sum_i \frac{\rho}{m_{\rm u}} x_i}{\sum_i \frac{\rho x_i}{m_{\rm u} A_i} + n_e}, \tag{5.5}$$

or

$$\mu = \frac{\sum_i \frac{\rho}{m_{\rm u}} x_i}{\sum_i \frac{\rho x_i}{m_{\rm u} A_i}(1+Z_i)}. \tag{5.6}$$

- One cleaner way of writing this is

$$\mu^{-1} = \frac{\sum_i x_i/A_i(1+Z_i)}{\sum_i x_i} = \sum_i \frac{x_i}{A_i}(1+Z_i). \tag{5.7}$$

- For example, for a neutral gas ($Z = 0$) we have

$$\mu^{-1} = \sum_i \frac{x_i}{A_i} \approx \left(X + \frac{Y}{4} + \frac{Z}{\overline{A}_i}\right)^{-1}, \tag{5.8}$$

where it is standard to write mass fractions X for hydrogen, Y for helium, and Z for everything else (metals), where $X + Y + Z = 1$.

- $\overline{A}_i$ is an average over metals, which at solar composition is about 15.5.

- For a fully ionized gas

$$\mu^{-1} \approx \sum_i \frac{x_i}{A_i}(1+Z_i) \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z, \tag{5.9}$$

or

$$\mu \approx \frac{4}{3 + 5X - Z}, \tag{5.10}$$

where for metals we usually approximate $(1 + Z_i)/A_i \approx 1/2$ (roughly equal number of protons and neutrons). We've eliminated Y in this expression through $Y = 1 - X - Z$.

IN CLASS WORK

Compute the mean molecular weight for (1) the ionized solar photosphere, where we have 90% hydrogen, 9% helium, and 1% heavy elements by mass, (2) the ionized solar interior where 71% hydrogen, 27% helium, and 2% heavy elements by mass, (3) completely ionized hydrogen, (4) completely ionized helium, and finally (5) neutral gas at the solar interior abundance.

Answer: (1) For the photosphere we can write

$$\mu^{-1} = 0.9\frac{2}{1} + 0.09\frac{3}{4} + 0.01\frac{1}{2} = 1.8725,$$

or $\mu \approx 0.53$.
(2) For the interior we can write

$$\mu^{-1} = 0.71\frac{2}{1} + 0.27\frac{3}{4} + 0.02\frac{1}{2} = 1.63,$$

or $\mu \approx 0.61$.
(3) For hydrogen, we will take $X = Z = A = 1$, and find then that $1/\mu = 2$.
(4) For helium, $X = Z = 0$ and $Y = 1$, so $\mu = 4/3$.
(5) For a neutral gas, we have

$$\mu^{-1} = 0.71 + 0.27\frac{1}{4} + 0.02\frac{1}{15.5} = 0.779,$$

or $\mu \approx 1.28$.

- From the above, we can also consider separately the mean molecular weight for ions and electrons.
- For ions, define $\mu_{\rm I}$ as

$$n_{\rm I} = \frac{\rho}{\mu_{\rm I} m_{\rm u}}. \tag{5.11}$$

Recall that

$$n_{\rm I} = \sum_j n_{j,\rm I} = \frac{\rho}{m_{\rm u}} \sum_j \frac{x_j}{A_j}. \tag{5.12}$$

So that

$$\mu_{\rm I} = \left(\sum_j \frac{x_j}{A_j} \right)^{-1}. \tag{5.13}$$

- This result should make sense, since above in Equation (5.8) we did not consider electrons.
- For electrons it's a bit harder since not all electrons need be free. But we will still define the *mean molecular weight per election* $\mu_{\rm e}$:

$$n_{\rm e} = \frac{\rho}{\mu_{\rm e} m_{\rm u}} \tag{5.14}$$

- Fully ionized, each atom contributes Z electrons. If an ion is partially ionized, we can consider the fraction yZ. (To compute the proper fraction of ionization of a gas (n_e), one needs to use the *Saha equation*).
- As before then

$$n_e = \sum_j n_{e,j} = \sum_j n_{j,\rm I} y_j Z_j = \frac{\rho}{m_{\rm u}} \sum_j \left(\frac{x_j}{A_j} \right) y_j Z_j, \tag{5.15}$$

which defines

$$\mu_e = \left(\sum_j \frac{x_j y_j Z_j}{A_j}\right)^{-1}. \tag{5.16}$$

- So finally

$$n = n_e + n_{\rm I} = \frac{\rho}{\mu m_{\rm u}}, \tag{5.17}$$

where

$$\mu = \left(\frac{1}{\mu_{\rm I}} + \frac{1}{\mu_e}\right)^{-1}. \tag{5.18}$$

IN CLASS WORK

Compute an expression for $\mu_{\rm e}$ in the deep stellar interior as a function only of X. Ignore metals.

Answer: Fully ionized case. We can write

$$\begin{aligned}\mu_{\rm e} &\approx \left(\frac{1}{1}X + \frac{2}{4}Y\right)^{-1} \\ &= \left(X + \frac{1}{2}(1-X)\right)^{-1} \\ &= \left(\frac{X+1}{2}\right)^{-1} = \frac{2}{1+X}.\end{aligned}$$

This should make sense. For a full H gas, the mean mass of particles per number of electrons (1/1) is 1. For a He gas ($X = 0$), we have a mass of 4 divided by 2 electrons, or $\mu_e = 2$.

Unit 6

Equation of state: Ideal gas

6.1 Preliminaries

- First we recall the distribution function and do a little thermodynamics with them.
- A distribution function simply measures the number density of a species in 6D space of position and momentum.
- If we know this function for a gas, all thermodynamic quantities can be derived (pressure, temperature, density, composition).
- Equations of state relate pressure, temperature, and number of particles.
- An ideal gas is one in which the particles don't interact (except through elastic collisions). They can exchange energy though, but have to conserve it.
- This approximation breaks down when matter is degenerate, and particles begin to "sense" each other and interact in quantum fashion or otherwise.
- An important thermodynamical quantity is the chemical potential μ_c for each species, that was introduced earlier. For classical particles, $\mu_c \to -\infty$, for degenerate fermions $\mu_c \to \epsilon_{\mathrm{F}}$, for bosons $\mu_c = 0$.
- Chemical changes in the gas use the chemical potential to monitor particle numbers, and thus to achieve a chemical equilibrium (in addition to thermodynamic equilibrium).
- In thermodynamic equilibrium, statistical mechanics tells us the relationship between the number density (of phase space, ie, number per unit volume per unit momentum: $\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}$) of a species

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left\{\left[E_j + E(p) - \mu_c\right]/k_{\mathrm{B}}T\right\} \pm 1}, \tag{6.1}$$

  where

  - j are the possible energy states of the species (like energy levels in an ion), and E_j is the energy of that level
  - $E(p)$ is the kinetic energy
  - g_j is the degeneracy of state j (number of states with same energy)
  - $\pm$ is either for fermions or bosons, respectively.

  Note that the units of Planck's constant are length times momentum (remember the uncertainty principle). We will come back to this frequently.

- To find the number density (particles cm^{-3}) we integrate $n(p)\,\mathrm{d}^3p$ over momentum space (assumed to be symmetric)

$$n = \int_p 4\pi p^2 n(p)\,\mathrm{d}p. \tag{6.2}$$

- To remain completely general, the kinetic energy of a particle of rest mass m is

$$E(p) = (p^2c^2 + m^2c^4)^{1/2} - mc^2. \tag{6.3}$$

IN CLASS WORK

What does this expression reduce to in the nonrelativistic limit?

Answer: In this limit, we note that $pc \ll mc^2$, so one can expand the term in the square root:

$$\begin{aligned} E(p) &= mc^2\left(1 + \frac{p^2c^2}{m^2c^4}\right)^{1/2} - mc^2, \\ &\approx mc^2\left(1 + \frac{1}{2}\frac{p^2c^2}{m^2c^4}\right) - mc^2, \\ &\approx \frac{p^2}{2m}, \end{aligned}$$

which is the expression we'd expect.

- Now we can define three general quantities:
  1. Velocity:
  $$v = \frac{\partial E}{\partial p}. \tag{6.4}$$
  2. Pressure:
  $$P = \int_{\boldsymbol{p}} n(\boldsymbol{p})\boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{p} = \frac{1}{3}\int_p n(p)v\,p\,4\pi p^2\,\mathrm{d}p, \tag{6.5}$$
  where the last equality comes from assuming isotropy of pressure.
  3. Internal energy density (energy per unit volume):
  $$u = \int_p n(p)E(p)4\pi p^2\,\mathrm{d}p. \tag{6.6}$$

- These general considerations can soon be applied to specific cases.

6.2 Maxwell-Boltzmann statistics

- The relation of Equation (6.1) to classical probability functions is found through Equation (6.11) and Equation (6.12). These tell us the fraction of particles within an infinitesimal element of 3-dimensional space (velocity, energy, or momentum space).

- In momentum space

$$f(p)\,\mathrm{d}p = \frac{4\pi}{(2\pi m k_\mathrm{B} T)^{3/2}} e^{-p^2/2mkT} p^2\,\mathrm{d}p. \tag{6.7}$$

- In energy space

$$f(E)\,\mathrm{d}E = \frac{2}{\sqrt{\pi}(k_\mathrm{B}T)^{3/2}} e^{-E/kT}\sqrt{E}\,\mathrm{d}E. \tag{6.8}$$

- In velocity space

$$f(v)\,\mathrm{d}v = 4\pi\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} e^{-mv^2/2kT}v^2\,\mathrm{d}v. \tag{6.9}$$

- These are normalized such that the integrals of each quantity over velocity, momentum, or energy are equal to 1.

6.3 Ideal monatomic gas

- As a first demonstration, we consider a gas of single species nonrelativistic particles. We will be using Equation (6.1).
- Their energy is $E = p^2/2m$. Consider one energy level $E_j = E_0$.
- For this system, the chemical potential goes to negative infinity (as we'll see), so the exponential term is large, and the ± 1 term can be safely ignored.
- The number density of particles in any given momentum state p is

$$n(p) = \frac{g}{h^3}e^{-p^2/2mkT}e^{-E_0/kT}e^{\mu_{\mathrm{c}}/kT}, \tag{6.10}$$

and so the total number density over all momenta is

$$n = \frac{4\pi g}{h^3}\int_0^\infty p^2 e^{-p^2/2mkT}e^{-E_0/kT}e^{\mu_{\mathrm{c}}/kT}\,\mathrm{d}p. \tag{6.11}$$

- The integral is straightforward and gives an expression

$$n = \frac{(2\pi m k_{\mathrm{B}}T)^{3/2}g}{h^3}e^{-E_0/kT}e^{\mu_{\mathrm{c}}/kT}. \tag{6.12}$$

- Another way to write this is

$$e^{\mu_{\mathrm{c}}/kT} = \frac{nh^3}{g(2\pi m k_{\mathrm{B}}T)^{3/2}}e^{E_0/kT}. \tag{6.13}$$

Since we are assuming that the term on the left is small (since $\mu_{\mathrm{c}} \ll -1$), then the right hand side must also be small. Specifically, $nT^{-3/2}$ cannot be too large. If that were the case, then we would not be able to ignore the ± 1 term in the distribution function.

- Returning to the definition of gas pressure in Equation (6.5), we can compute the integral to find

$$P = g\frac{4\pi}{h^3}\frac{\pi^{1/2}}{8m}(2mk_{\mathrm{B}}T)^{5/2}e^{-E_0/kT}e^{\mu_{\mathrm{c}}/kT}. \tag{6.14}$$

- Using the generalized number density from Equation (6.12), this gives what you thought it would

$$P = nk_{\mathrm{B}}T\ [\mathrm{dyne\,cm^{-2}}]. \tag{6.15}$$

This is the equation of state for an ideal gas.

- Similarly we can compute the internal energy density from Equation (13.7)

$$u = \frac{3}{2}nk_{\mathrm{B}}T = \frac{3}{2}P;\ [\mathrm{erg\,cm^{-3}}]. \tag{6.16}$$

- Note that the units of pressure and internal energy density are the same.

- From what we saw before with the mean molecular weight, we can also express these quantities as

$$n = \frac{\rho}{\mu m_{\mathrm{u}}}, \tag{6.17}$$

$$P = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}}, \tag{6.18}$$

$$P = \frac{\rho R T}{\mu}, \tag{6.19}$$

where μ is the mean molecular weight, and $R = k_{\mathrm{B}}/m_{\mathrm{u}}$ is the ideal gas constant $R = 8.31 \times 10^7\,\mathrm{erg\,K^{-1}\,mol^{-1}}$.

EXAMPLE PROBLEM 6.1: Instead of arriving at Equation (6.16) through the energy formulation, one can use velocity to show that the average internal kinetic energy per particle is $3/2k_{\mathrm{B}}T$. Hint: Start with the Maxwellian distribution for a classical gas in velocity (Equation (6.9)) and then compute the mean square speed $\langle v^2 \rangle$. The integration limits of v should be from zero to infinity.

Answer: The mean square speed can be written as

$$\langle v^2 \rangle = \int_0^\infty v^2 f(v) \mathrm{d}v,$$

where $f(v)$ is the Maxwell distribution. One can (and should) use a table of integrals to find that

$$\int_0^\infty x^n e^{-ax^2} \mathrm{d}x = \frac{(2k-1)!!}{2^{k+1} a^k} \left(\frac{\pi}{a}\right)^{1/2},$$

where $n = 2k$ and in our case $a = m/2k_{\mathrm{B}}T > 0$. Note the double factorial, which, for $k = 2$, is $3 \times 1 = 3$. The result is

$$\langle v^2 \rangle = 4\pi \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \cdot \frac{3}{8} \frac{\pi^{1/2}}{a^{5/2}},$$

and after cancellation becomes

$$\langle v^2 \rangle = \frac{3k_{\mathrm{B}}T}{m}.$$

The kinetic energy is then $1/2m\langle v^2 \rangle = 3/2k_{\mathrm{B}}T$, precisely what we set out to prove.

Unit 7

Equation of state: Degenerate gas

7.1 Completely degenerate gas

- The ideal gas law (because of Maxwell-Boltzmann statistics) breaks down at sufficiently high densities and/or low temperatures.
- Consider the extreme case where $T \to 0$ at fixed density.
- From Equation (6.7), the Maxwell distribution peaks at zero momentum where all the particles want to pile up. They want to be in the lowest energy state, which is zero.
- There's a limit to how close fermions can come, based on the Pauli exclusion principle.
- So instead, we must use Fermi-Dirac statistics and not Maxwell-Boltzmann.
- Consider first the most interesting terms in Equation (6.1)

$$f(p) = \frac{1}{\mathrm{e}^{(E(p)-\mu_c)/kT} + 1}, \tag{7.1}$$

  where we've taken a reference energy level $E_j = 0$.
- As $T \to 0$, f goes to 1 or 0 depending on the sign of $E - \mu_c$.
- The function is discontinuous at the Fermi momentum p_F, or at energy E_F.
- For fermions such as electrons, with spin 1/2, the degeneracy factor in Equation (6.1) $g = 2$.
- The chemical potential is the Fermi energy $\mu_c = E_F$, up to which all the quantum states are filled. This is what is meant by "degeneracy."
- It is convenient to introduce the dimensionless momentum $x = p/mc$ and Fermi momentum $x_F = p_F/mc$.
- The integration to obtain the number density of electrons is therefore

$$n_e = \frac{8\pi}{h^3}\int_0^{p_F} p^2\,\mathrm{d}p = 8\pi\left(\frac{h}{m_e c}\right)^{-3}\int_0^{x_F} x^2\,\mathrm{d}x = \frac{8\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} x_F^3 = 5.865\times10^{29}\,x_F^3\ \mathrm{cm}^{-3}. \tag{7.2}$$

  Note that $p_F \sim n_e^{1/3}$.
- It is common, yet confusing, to remove the subscript for Fermi and just to write

$$n_e = 5.865\times10^{29}\,x^3\ \mathrm{cm}^{-3}. \tag{7.3}$$

- If we reintroduce the electron mean molecular weight, as in Equation (5.14), we can get this in terms of mass density
$$\frac{\rho}{\mu_e} = \frac{8\pi m_{\mathrm{u}}}{3}\left(\frac{h}{m_e c}\right)^{-3} x_{\mathrm{F}}^3 = 9.74 \times 10^5\, x_{\mathrm{F}}^3 \text{ g cm}^{-3}. \tag{7.4}$$
This is interesting in that it is a way to determine the Fermi momentum or energy if provided a value for ρ/μ_e. Also note the densities of about 10^6, which are typical of white dwarfs.

- The electron pressure (the pressure due to degenerate electrons, not ions) from Equation (6.5) is therefore
$$P_e = \frac{8\pi}{3}\frac{m_e^4 c^5}{h^3}\int_0^{x_{\mathrm{F}}} \frac{x^4}{(1+x^2)^{1/2}}\mathrm{d}x = Cf(x), \tag{7.5}$$
where $C = \pi m_e^4 c^5/3h^3 = 6.002 \times 10^{22}\,\mathrm{dyne\,cm^{-2}}$.

- The function $f(x)$ is
$$f(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\sinh^{-1} x. \tag{7.6}$$

- Similarly, the internal energy density from Equation (13.7) is
$$u_e = 8\pi\frac{m_e^4 c^5}{h^3}\int_0^{x_{\mathrm{F}}} x^2\left[(1+x^2)^{1/2}-1\right]\mathrm{d}x = Cg(x), \tag{7.7}$$
where C is the same as before.

- The function $g(x)$ is
$$g(x) = 8x^3\left[(1+x^2)^{1/2}-1\right] - f(x). \tag{7.8}$$

- These are very general expressions in terms of x for the pressure and energy density.

IN CLASS WORK

Show that x discriminates between the nonrelativistic regime $x \ll 1$ and the relativistic regime $x \gg 1$. Use Equation (6.3) and Equation (6.4).

Answer: Since $x = p/m_e c$, we need to know what the momentum is. It's not simply $p = mv$. We can compute the velocity and then the momentum.

$$v = \frac{\partial E}{\partial p} = \frac{p/m}{\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}}$$

Solving for p gives

$$p = \frac{mv}{\sqrt{1-v^2/c^2}},$$

which is expected. Therefore,

$$x = \frac{v/c}{\sqrt{1-v^2/c^2}}.$$

Another useful form is to solve for the velocity ratio in terms of x:

$$\frac{v^2}{c^2} = \frac{x^2}{1+x^2}.$$

In any case, for electron velocities near the speed of light, x becomes very large.

- Let us first consider the case for **nonrelativistic electrons** where $x \ll 1$. In this limit, to first order

$$f(x) \approx \frac{8}{5}x^5,$$
$$g(x) \approx \frac{12}{5}x^5.$$

- Using Equation (7.5) we thus have

$$P_e = \frac{8\pi m_e^4 c^5}{15h^3}x^5. \tag{7.9}$$

- Relating this through the density in Equation (7.2) to remove x, and then using the electron mean molecular weight in Equation (5.14), we arrive at the final expression

$$P_\mathrm{e} = 1.0036 \times 10^{13} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3} \mathrm{dyne\,cm^{-2}}. \tag{7.10}$$

  This is the equation of state for a fully degenerate nonrelativistic electron gas.

- Carrying out the same exercise for the internal energy, we find

$$u_\mathrm{e} = \frac{3}{2}P_\mathrm{e}. \tag{7.11}$$

- Thus, the equation of state for this nonrelativistic gas has characteristics of an ideal monatomic gas (see Equation (6.16)).

- Let us now consider the case for **relativistic electrons** where $x \gg 1$. In this limit, to first order

$$f(x) \approx 2x^4,$$
$$g(x) \approx 6x^4.$$

- Now, the pressure is

$$P_\mathrm{e} = \frac{2\pi m_e^4 c^5}{3h^3}x^4, \tag{7.12}$$

  which after plugging in constants and introducing n_e and ρ gives

$$P_\mathrm{e} = 1.243 \times 10^{15} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3} \mathrm{dyne\,cm^{-2}}. \tag{7.13}$$

- Similarly for the energy we have

$$u_\mathrm{e} = 3P_\mathrm{e}. \tag{7.14}$$

- The transition from non- to relativistic states is smooth in x. Note the exponents on the density, which we will come back to these later (polytropes).

PROBLEM 7.1: [10 pts]: Find the ratio of the electron degeneracy pressure to the electron ideal gas pressure in the center of the (current) Sun (assuming the center could be degenerate in electrons). Let $T = 15 \times 10^6$ K, $\rho = 150\,\mathrm{g\,cm^{-3}}$, and abundances $X = 0.35$, $Z = 0.02$. Prove that you are using the correct expression for the electron degeneracy pressure.

Answer: The gas pressure of the electrons is

$$P_\mathrm{e} = \frac{\rho k_\mathrm{B} T}{\mu_e m_\mathrm{u}}.$$

The mean molecular weight per electron is given in Equation (5.16). Here it is

$$\mu_e = (X + Y/2 + Z/2)^{-1} = (0.35 + (0.5)(0.63) + (0.5)(0.02))^{-1} = 1.48.$$

Then we have

$$P_e = \frac{(150)(1.38 \times 10^{-16})(15 \times 10^6)}{(1.48)(1.66 \times 10^{-24})} = 1.26 \times 10^{17}\,\mathrm{dyne\,cm^{-2}}.$$

Are the degenerate electrons relativistic or nonrelativistic? Equation (7.4) tells us. Solving for x gives $x = 0.05$, which is well within the nonrelativistic limit. Therefore, the degenerate electrons provide a pressure of

$$P_e^{\mathrm{deg}} = 1.0036 \times 10^{13} \left(\frac{150}{1.48}\right)^{5/3} = 2.2 \times 10^{16}\,\mathrm{dyne\,cm^{-2}}.$$

The ratio is therefore about 0.175. If we used the ions too in the gas pressure calculation, the ratio would be smaller.

7.2 Partially degenerate gas

- The previous section considered an ideal zero-temperature gas. But if the temperature is finite, then the Fermi-Dirac function is not a simple step function and needs to be evaluated numerically.
- When this is done, the temperature dependence of the equation of state is realized.
- The typical expressions are just expansions of the Fermi function in powers of T.
- Qualitatively though, as temperature is increased some amount, only the electrons near the Fermi energy have the freedom to move to higher states and smear out the step function. Only if a temperature equivalent to about the Fermi energy $E_F = k_B T$ is achieved can particles deep within the Fermi sea find unoccupied levels at higher energies. If that's the case, the gas becomes more like a classical one.
- So the transition from degeneracy to non degeneracy can roughly be considered to occur when the temperature of the gas is near the Fermi energy.
- We will not spend more time on this right now, but keep in mind that these states of matter are not typically homogeneous, but rather a mixture.

Unit 8

Density-temperature equation of state landscape

8.1 Radiation pressure

First consider one more equation of state.

- Particles are not the only source of pressure in a star. The radiation field of photons can also exert a pressure.
- Photons have momentum and can exchange that momentum with other objects, creating pressure.
- We already have an expression for the general pressure in Equation (6.5).
- With a degeneracy factor $g = 2$ (photons have 2 spin states, or polarizations, each with the same energy at fixed frequency), chemical potential (bosons) $\mu_c = 0$, $E = pc$, and $E_j = 0$, the distribution function Equation (6.1) is
$$n(p) = \frac{2}{h^3} \frac{1}{\exp\left(pc/kT\right) - 1}. \tag{8.1}$$
- From Equation (6.5), we thus have
$$P_{\rm rad} = \frac{8\pi c}{3h^3} \int_0^\infty \frac{p^3}{{\rm e}^{pc/kT} - 1} \, {\rm d}p, \tag{8.2}$$
where we made the substitution $v \equiv c$ for photons.
- The integral can be solved (Problem 8.1) to give
$$P_{\rm rad} = \frac{1}{3} a \, T^4, \tag{8.3}$$
where $a = 4\sigma/c = 7.5 \times 10^{-15} \, {\rm erg \, cm^{-3} \, K^{-4}}$.
- Similarly as before, the energy density
$$u_{\rm rad} = aT^4 = 3P_{\rm rad}. \tag{8.4}$$
Note the similar form to relativistic, degenerate matter.

PROBLEM 8.1: [10 pts]: Carry out the integral in Equation (8.2) to show that

$$a = \frac{8\pi^5}{15}\frac{k^4}{h^3c^3}. \tag{8.5}$$

Hint: make the substitution $x = pc/kT$.

Answer: Upon making the suggested subsitution, Equation (8.2) becomes

$$P_{\rm rad} = \frac{8\pi c}{3h^3}\left(\frac{kT}{c}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\,dx.$$

The solution to an integral of this form is

$$\text{Integral} = \Gamma(4)\zeta(4),$$

(see http://dlmf.nist.gov/25.5 for example) where the gamma function

$$\Gamma(4) \equiv (4-1)! = 6,$$

and the Riemann zeta function

$$\zeta(4) = \frac{\pi^4}{90}.$$

Putting this all together with the prefactors gives you the desired constant a.

8.2 Putting it all together

- Putting the previous sections together, the pressure of stellar matter through the equation of state in general is

$$P = P_{\rm ion} + P_{\rm e} + P_{\rm rad}. \tag{8.6}$$

- In some cases, the electron pressure is from degenerate particles. In rare cases, the ions can become degenerate too.
- Not all of these pressure terms contribute equally to the total pressure at any given time, as you saw in Problem 7.1.
- Consider the total gas pressure of an ideal gas as

$$P_{\rm gas}^{\rm ideal} = P_{\rm ion} + P_{\rm e} = \frac{\rho k_{\rm B} T}{\mu m_{\rm u}}. \tag{8.7}$$

- It is useful to compare regions where this and $P_{\rm e}^{\rm deg}$ and $P_{\rm rad}$ compete and transition to one another.
- First consider where an ideal gas transitions to a degenerate nonrelativistic one. Equating Equation (7.10) and Equation (8.7) gives

$$\frac{\rho}{\mu_{\rm e}^{5/2}} = \left(\frac{k_{\rm B}}{C\mu m_{\rm u}}\right)^{3/2} T^{3/2}, \tag{8.8}$$

  where C is the constant prefactor in Equation (7.10) .
- For large densities or low temperatures, i.e., when $\rho T^{-3/2} >$ const, the gas is dominated by degenerate pressure. This is shown by the line of a slope of 2/3 in Figure 8.1.

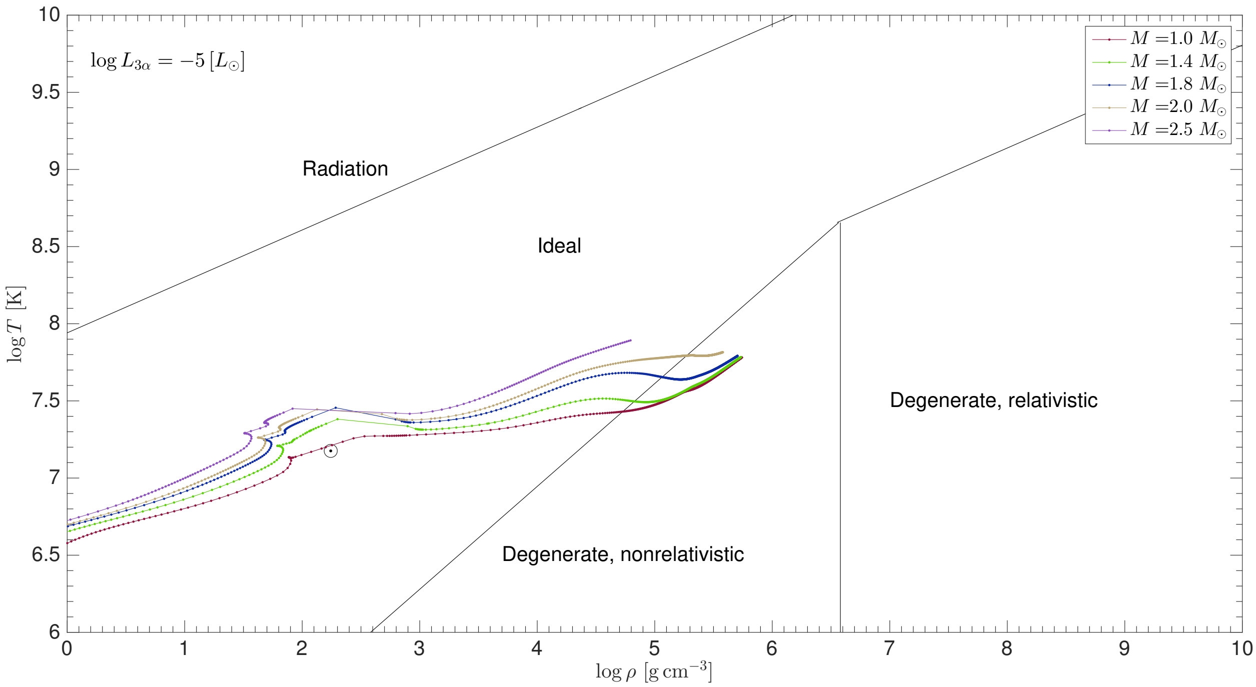

Figure 8.1: Stellar matter conditions. These boundary regimes are computed using $\mu_\mathrm{e} = 2$ and $\mu = 0.5$. A few example model tracks (central values as a funciton of time) are shown for different masses, until the point when the contribution to the luminosity by the triple$-\alpha$ process is 1 part in 10^5 solar luminosities, as denoted in the figure. In practice, these stars are near their respective tip of the red-giant branch. The current location of the Sun is given by its symbol. The radiation pressure boundary is computed from $P_\mathrm{rad} = 10 P_\mathrm{ideal}$.

- When electron speeds become relativistic, Equation (7.13) becomes appropriate, and when equated with the ideal gas pressure yields
$$\frac{\rho}{\mu_\mathrm{e}^4} = \left(\frac{k_\mathrm{B}}{D\mu m_\mathrm{u}}\right)^3 T^3, \tag{8.9}$$
where D is the constant prefactor in Equation (7.13).

- This is shown in Figure 8.1 with the line of slope 1/3 at high temperature and density.

- In the degenerate regime, the transition from non- to relativistic is found by equating Equation (7.10) and Equation (7.13), which is independent of temperature (since these are completely degenerate systems):
$$\frac{\rho}{\mu_\mathrm{e}} = \left(\frac{D}{C}\right)^3. \tag{8.10}$$
This is given in the figure by the vertical line.

- Finally, we can determine where radiation pressure starts to exceed ideal gas pressure. We use Equation (8.3) to find
$$\frac{\rho}{\mu} = \frac{1}{3}\frac{a m_\mathrm{u}}{k_\mathrm{B}} T^3. \tag{8.11}$$

- In Figure 8.1 this is shown by the line at the bottom right of slope 1/3.

Unit 9

Hydrostatic Equilibrium

9.1 Derivation

- Consider Figure 9.1. Take a thin mass element in a star of thickness $\mathrm{d}r$ and surface $\mathrm{d}A$ at radius r (and thus of mass $\mathrm{d}m = \rho \mathrm{d}r \mathrm{d}A$) from the center.
- The gravitational force on that mass element is

$$\mathrm{d}F_g = -\frac{G\left[\rho(r)\mathrm{d}r\mathrm{d}A\right] m(r)}{r^2}, \tag{9.1}$$

  directed radially inward.
- In equilibrium, this force is balanced by an outward pressure force acting at r and $r+\mathrm{d}r$ ($P = \mathrm{d}F/\mathrm{d}A$)

$$\mathrm{d}F_P = \left[P(r) - P(r+\mathrm{d}r)\right]\mathrm{d}A = -\frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r\mathrm{d}A, \tag{9.2}$$

  by using the definition of the derivative.
- In hydrostatic equilibirum, $\mathrm{d}F_g + \mathrm{d}F_P = 0$, and so

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2}. \tag{9.3}$$

- We will commonly see this written in vector form as

$$\boldsymbol{\nabla}_{\boldsymbol{r}} P = \rho \boldsymbol{g}. \tag{9.4}$$

- Note that the mass element in the thin shell can be expressed as

$$\mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r, \tag{9.5}$$

  or

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2. \tag{9.6}$$

- So another convenient form is

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}m} = -\rho g \frac{1}{4\pi r^2 \rho} = -\frac{Gm}{4\pi r^4} \tag{9.7}$$

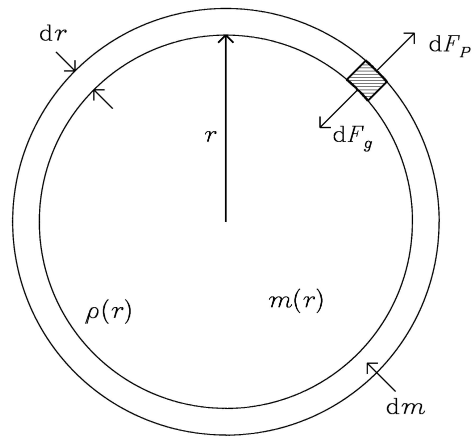

Figure 9.1: Schematic for deriving hydrostatic equilibrium. From Christensen-Dalsgaard [2003].

- Thus the mass as a function of radius is found by

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\mathrm{d}r'. \tag{9.8}$$

IN CLASS WORK

Estimate the central pressure of the Sun from the equation of hydrostatic equilibrium. Compare to the tabulated values. Try to put all final expressions in terms of scaled solar values as we have been doing.

Answer: The simplest thing one can do is ignore the derivatives in the equilibrium expression and assume the density is constant:

$$\frac{\partial P}{\partial r} = -\frac{G\rho(r)M(r)}{r^2} \tag{9.9}$$

$$\frac{P_c}{R} = \frac{3}{4\pi}\frac{GM^2}{R^5} \tag{9.10}$$

$$P_c = \frac{3}{4\pi}\frac{GM_\odot^2}{R_\odot^4}, \tag{9.11}$$

where we've replaced all values by the gross solar ones. For a general star we can show

$$P_c \approx 2.69 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \mathrm{dyne\,cm^{-2}}.$$

We know that 1 dyne cm^{-2} $= 0.1$ N m^{-2} [Pa], so we are only off in pressure by about 2 orders of magnitude when compared to the tabulated value of $2.3 \times 10^{17}\,\mathrm{dyne\,cm^{-2}}$. Not so bad actually. The tabulated value is roughly

$$P_c \approx \frac{261}{4\pi}\frac{GM_\odot^2}{R_\odot^4} \quad \text{(tabulated)}$$

PROBLEM 9.1: [10 pts]: (a) Do the same type of calculation as for the central pressure to find the central temperature and compare to the table value. Use the same mass fractions as in Problem 7.1. (b) Then, using the expressions you now have for the gas pressure and temperature, show that we can ignore the radiation pressure for the Sun in the core. I.e., show that $P_{\rm R}/P_{\rm G} \approx c(M/M_\odot)^2$, where c is a smallish number. Try to put all final expressions in terms of scaled solar values as we have been doing.

Answer: (a) Use the equation of state:

$$
\begin{aligned}
T_c &= \frac{P_c \mu m_{\rm u}}{\rho k_{\rm B}} \\
T_c &= \frac{\mu m_{\rm u}}{\rho k_{\rm B}} \cdot \frac{3}{4\pi} \frac{GM^2}{R^4} \\
T_c &= \frac{\mu m_{\rm u}}{k_{\rm B}} \frac{R^3}{M} \frac{GM^2}{R^4} \\
T_c &= \frac{\mu m_{\rm u} GM}{k_{\rm B} R}.
\end{aligned}
$$

The mean molecular weight is

$$\mu = [2X + 3Y/4 + Z/2]^{-1} = [(2) * (0.35) + (0.75)(0.63) + (0.5)(0.02)]^{-1/2} = 0.846.$$

Using the relevant solar values, this all gives

$$T_c = 1.94 \times 10^7 \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1} \text{ K}.$$

The temperature is a bit smaller than the tabulated value of 1.57×10^7 K, but again, not bad.
(b) Now we want to find the ratio $P_{\rm R}/P_{\rm G}$. We have just found the 3 expressions:

$$
\begin{aligned}
P_{\rm R} &= \frac{1}{3} a T^4 \\
P_{\rm G} &= \frac{3}{4\pi} \frac{GM^2}{R^4} \\
T_c &= \frac{G \mu m_{\rm u} M}{k_{\rm B} R}.
\end{aligned}
$$

We know that $a = 7.5 \times 10^{-15}$ erg cm^{-3} K^4. Plugging in we can show

$$
\begin{aligned}
\frac{P_{\rm R}}{P_{\rm G}} &= \frac{1}{3} a G^3 \left(\frac{\mu m_p}{k_{\rm B}}\right)^4 \frac{4\pi}{3} M_\odot^2 \left(\frac{M}{M_\odot}\right)^2 \\
&\approx 0.13 \left(\frac{M}{M_\odot}\right)^2.
\end{aligned}
$$

So in general only for t massive stars is the radiation pressure important (typically starting about $50\,\mathrm{M}_\odot$).

EXAMPLE PROBLEM 9.1: In Equation (9.11) we found a cheap and dirty estimate of the central pressure. Now, using Equations (9.3) and (9.6), we can find a **lower bound** for the pressure at the center of the Sun. First we need an expression for $\mathrm{d}P/\mathrm{d}m$ and then integrate from core to surface. Making a simple assumption in the integrand allows you to argue this is really a lower limit. Compare again to the previous result of the in class problem by expressing your final answer in terms of

$$P_c = \text{const} \times \frac{GM_\odot^2}{R_\odot^4}, \tag{9.12}$$

where the constant is really the key quantity in your computation.

Answer:

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}.$$

We can take simply that $r = R_\odot$. Integrate both sides from the center to the surface:

$$\int_{m=0}^{m=M_\odot} \mathrm{d}P = P(M_\odot) - P(0) = -\int_{m=0}^{m=M_\odot} \frac{Gm}{4\pi r^4}\,\mathrm{d}m.$$

To obtain an absolute lower limit, let's just take $r = R_\odot$, it's the simplest thing we can do and we know it makes the overall value the smallest. Then

$$P(M_\odot) - P(0) = -\frac{G}{4\pi R_\odot^4}\int_{m=0}^{m=M_\odot} m\,\mathrm{d}m = \frac{1}{8\pi}\frac{GM_\odot^2}{R_\odot^4} < P(0).$$

This gives

$$P_c \approx 4.5\times 10^{14}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4}\,\mathrm{dyne\,cm^{-2}},$$

below the value in the table and below the one found before. But it is a lower limit. It is very interesting that this limit is valid for any star in hydrostatic equilibrium, independent of equation of state or energy production.

EXAMPLE PROBLEM 9.2: Let's improve the lower limit now (i.e., make it a bit larger). All that is needed is to assume a mean density that is a decreasing function of r such as

$$\overline{\rho(r)} = \frac{m}{4\pi r^3/3}.$$

Use this at the right step in Problem 9.1 to get a new lower limit, again expressed as

$$P_c = \mathrm{const}\times\frac{GM_\odot^2}{R_\odot^4}.$$

Answer: Picking up where we left off by solving for r as a function of the density and mass gives

$$P(0) > \frac{G}{4\pi}\int_0^{M_\odot}\frac{m}{(3m/4\pi\overline{\rho})^{4/3}}\mathrm{d}m = \frac{G}{4\pi}\left(\frac{4\pi\overline{\rho}}{3}\right)^{4/3}\int_0^{M_\odot} m^{-1/3}\mathrm{d}m = \frac{G}{4\pi}\left(\frac{4\pi\overline{\rho}}{3}\right)^{4/3}\frac{3}{2}M_\odot^{2/3} = \frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^4}.$$

In the last step we simply allowed the mean density to be the gross solar values. This gives us a new value 3 times larger than the previous one, but still twice as small as that in Equation (9.11).

9.2 Simple solutions

We can solve the equation of hydrostatic equilibrium if we know the density as a function of radius or pressure. Analytically, we can make some progress from simplified assumptions.

9.2.1 Linearized density

- Let

$$\rho = \rho_c\left(1 - \frac{r}{R}\right). \tag{9.13}$$

- Then integrating Eq. 9.6 with this density profile gives

$$m(x) = \frac{4}{3}\pi R^3\rho_c(x^3 - \frac{3}{4}x^4), \tag{9.14}$$

where $x = r/R$.

- We know that it must be true that the total mass

$$M = m(x=1) = \frac{\pi}{3} R^3 \rho_c, \tag{9.15}$$

or

$$\rho_c = \frac{3M}{\pi R^3}. \tag{9.16}$$

- Using hydrostatic equilibrium and what we've just found, we can show

$$P = \frac{5}{4\pi} \frac{GM^2}{R^4} \left(1 - \frac{24}{5} x^2 + \frac{28/5}{x}^3 - \frac{9}{5} x^4 \right). \tag{9.17}$$

- Note how the coefficient is larger than before at the center ($x = 0$)

9.2.2 Isothermal atmosphere

- Take hydrostatic equilibrium in one dimension:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g. \tag{9.18}$$

- For an incompressible fluid (constant density)

$$P = P_0 - \rho g r, \tag{9.19}$$

where P_0 is the pressure at $r = 0$. An incompressible fluid at rest increases linearly with depth.

- Also, using the equation of state and an **isothermal** ideas gas, we can replace P in the hydrostatic equation with ρ:

$$\frac{RT}{\mu} \frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho g, \tag{9.20}$$

whose solution is

$$\rho = \rho_0 \exp\left(-\frac{r}{H} \right), \tag{9.21}$$

where the (density) scale height $H = RT/\mu g$.

Unit 10

Polytropes

10.1 Motivation and derivation

- So far we've collected 3 equations for stellar structure, collected here for convenience

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi\rho r^2, \\ \frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{G\rho m}{r^2}, \\ \frac{\mathrm{d}L}{\mathrm{d}r} &= 4\pi\rho r^2\,\varepsilon.\end{aligned}$$

- There are several others that we need to full-out model a real star. But even now we can get some very important insights on stellar structure.
- Ignore the 3rd equation for now. The first 2 equations have 3 unknowns and cannot be solved simultaneously as they stand.
- First law of thermodynamics

$$\frac{\mathrm{d}Q}{\mathrm{d}T} = \frac{\mathrm{d}U}{\mathrm{d}T} + P\frac{\mathrm{d}V}{\mathrm{d}T} = C = c_V + P\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_P. \tag{10.1}$$

- Ideal gas: $P = RT/V\mu$, where again $V = (1/\rho)$ is the specific volume and recall that $c_P - c_V = R/\mu$. We then find after manipulation

$$\frac{\mathrm{d}T}{T} + \frac{1}{n}\frac{\mathrm{d}V}{V} = 0, \tag{10.2}$$

  where n is the polytropic index $n = (c_V - C)/(c_P - c_V)$.
- We can eliminate the temperature from this to get pressure and density:

$$\frac{\mathrm{d}P}{P} = \left(1 + \frac{1}{n}\right)\frac{\mathrm{d}\rho}{\rho}, \tag{10.3}$$

  which, when integrated, gives

$$P = \mathrm{const} \times \rho^{(1+1/n)}. \tag{10.4}$$

- A system where pressure and density are related as $P = K\rho^{1+1/n}$ is called a polytrope.

- For an adiabatic change $C = 0$, $n = 1/(\gamma - 1)$, and
$$P = K\rho^{\gamma}, \tag{10.5}$$
where $\gamma = c_P/c_V$.

- This is very useful because we can now get radial profiles of $P(r), T(r), m(r)$ and $\rho(r)$.

10.2 Lane-Emden equation

- Take hydrostatic equilibrium, divide by density, multiply by r^2, and use the mass gradient equation:
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{10.6}$$

- Consider the polytropic equation of state
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 K\gamma\rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{10.7}$$

- Use polytropic index n and let the density be rewritten as a unitless quantity θ by
$$\frac{\rho}{\rho_c} = \theta^n, \tag{10.8}$$
where ρ_c is the central density of a model.

- Then
$$\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n. \tag{10.9}$$

- Let the coefficient (of units distance squared) be α^2, where
$$\alpha = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2} = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}. \tag{10.10}$$
(You might want to prove to yourself the above is true and the unit is distance).

- We then scale the radial coordinate
$$r = \alpha\xi. \tag{10.11}$$

- Then
$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{10.12}$$
This is known as the Lane-Emden equation.

- Boundary conditions at the center must satisfy (noting Equation (10.8))
$$\theta(\xi) = 1 \quad \text{at} \quad \xi = 0, \tag{10.13}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = 0 \quad \text{at} \quad \xi = 0. \tag{10.14}$$

- Define the surface as
$$\theta(\xi) = 0 \quad \text{at} \quad \xi = \xi_1. \tag{10.15}$$

10.3 Polytrope solutions

- In what follows, a subscript n denotes the label of the polytropic index, whereas the supercript n is the quantity raised to the nth power.
- Assume a solution $\theta_n(\xi)$ can be found to Equation (10.12) for a given index n.
- Then the radius of the model is

$$R = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2} \xi_1 = \alpha\xi_1. \tag{10.16}$$

- The mass interior to $m(\xi)$ is

$$\begin{aligned} m(\xi) &= \int_0^{\alpha\xi} 4\pi r'^2 \rho \, \mathrm{d}r' = 4\pi\alpha^3\rho_c \int_0^{\xi} \xi'^2\theta^n \, \mathrm{d}\xi' && (10.17)\\ &= -4\pi\alpha^3\rho_c \int_0^{\xi} \frac{\mathrm{d}}{\mathrm{d}\xi'}\left(\xi'^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi'}\right) \mathrm{d}\xi' && (10.18)\\ &= -4\pi\alpha^3\rho_c\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} && (10.19)\\ m(\xi) &= -4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n}\, \xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}. && (10.20) \end{aligned}$$

- The total mass is

$$M = -4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}. \tag{10.21}$$

- Inspecting the mass and radius relations, the constant K is

$$K = NGM^{(n-1)/n} R^{(3-n)/n}, \tag{10.22}$$

  where

$$N = \frac{(4\pi)^{1/n}}{n+1} \left[-\xi_1^{(n+1)/(n-1)} \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}\right]^{(1-n)/n}. \tag{10.23}$$

- It is interesting to point out here that if K is known from some equation of state, then one can derive explicit mass-radius relationships. If that is not the case, then mass and radius must be predefined. More about this later.
- Mean density of model

$$\overline{\rho} = \frac{M}{V} = -\frac{3}{\xi_1}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1} \rho_c. \tag{10.24}$$

- Central density is then

$$\rho_c = -\frac{\xi_1}{3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^{-1}_{\xi=\xi_1} \frac{M}{4/3\pi R^3}. \tag{10.25}$$

- Central pressure

$$P_c = K\rho_c^{(n+1)/n}, \tag{10.26}$$

  or

$$P_c = W_n \frac{GM^2}{R^4}, \tag{10.27}$$

where

$$W_n = \left[4\pi(n+1) \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right)^2_{\xi=\xi_1} \right]^{-1}. \quad (10.28)$$

Note that this is the coefficient we were computing in some simple example problems (cf. Equation (9.11)).

- So pressure throughout model is

$$P = P_c \theta^{n+1}. \quad (10.29)$$

- For the temperature, assume an ideal gas with μ constant, then

$$T = T_c \theta, \quad (10.30)$$

where

$$T_c = \Theta \frac{G M \mu m_{\mathrm{u}}}{k_{\mathrm{B}} R}, \quad (10.31)$$

and

$$\Theta = \left[-(n+1)\xi_1 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right)_{\xi=\xi_1} \right]^{-1}. \quad (10.32)$$

Again here, this coefficient is a number obtained through simpler means using only hydrostatic equilibrium (cf. Problem 9.1).

- The distribution of mass in a polytrope can be obtained easily (see Equation (10.20) and Equation (10.21))

$$q = \frac{m}{M} = \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right)^{-1}_{\xi=\xi_1}. \quad (10.33)$$

- Only 3 analytical solutions of the Lane-Emden equation are possible:

$$n = 0 \quad ; \qquad \theta_0 = 1 - \frac{1}{6}\xi^2, \quad (10.34)$$

$$n = 1 \quad ; \qquad \theta_1 = \frac{\sin \xi}{\xi}, \quad (10.35)$$

$$n = 5 \quad ; \qquad \theta_5 = (1 + \frac{1}{3}\xi^2)^{-1/2}. \quad (10.36)$$

COMPUTER PROBLEM 10.1: [40 points]: Solve the Lane-Emden Equation (10.12) numerically with the two boundary conditions using your method of choice for your assigned index n.

What to do

- To solve the 2nd-order nonlinear differential equation, which is a difficult task to do as is, the first thing you will need to do is to make a substitution to generate 2 first-order differential equations, as can always be done. A suitable choice may be to let $y = \theta$ and $z = \mathrm{d}\theta/\mathrm{d}\xi = y'$. You'll then be able to have an equation for y' and one for z'.
- Next you need to choose your solver. You can treat this as a boundary-value problem, in which case a method like Newton-Rhapson can be used. Fancy software like `IDL`, Python, and `MATLAB` have built-in boundary value algorithms, but can be tricky to implement in some cases. Perhaps a better option is to use such languages and code your own algorithm, perhaps implementing a "shooting method," which treats the problem as an initial-value problem. You "shoot" from the center, say, and work your way out to the end of the model using an integrator. A common and very handy integrator is Runge-Kutta. I'd suggest this method (shooting) because you don't need any sophisticated algorithms and it works! But it's your choice. A 4th-order Runge-Kutta is sufficient for this problem if you choose to do so. There's lots of information on this in the *Numerical Recipes* book, for example. If you do this, make

sure you verify through testing that you choose the proper grid spacing.

- You have to be a little careful about how you treat the center of the model since there is a possible divergence in your equation(s) (as should be apparent already). If one expands the Lane-Emden equation about the origin, it can be shown that

$$\theta_n(\xi) \simeq 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{15120}\xi^6 + \dots \tag{10.37}$$

You can use this approximation to set your first values for y and z (if you choose to solve the problem with this class of methods) by taking a very small, but finite, starting ξ. Then just run it until you cross the first zero in θ_n. Make sure your grid is sufficiently fine so the solution is smooth.

What to hand in

(a) You will solve for a polytrope of given index n, assigned in the table. You will provide the instructor the values (including 3 decimal places!) you find for the table columns for your n only. For the last 3 columns in the table, compute those values for 1 solar mass, 1 solar radius, and composition $X = 0.7$ and $Z = 0.02$. We'll fill the table in together when finished. Also provide a copy of your code. [15 pts]

(b) Plot the following quantities on a single full-page plot with appropriate labels and clearly distinguishable lines: $\theta_n(\xi)$, $\theta_n^n(\xi)$, $\theta_n^{n+1}(\xi)$, and $q(\xi)$. What do each of the 4 quantities that you are showing here physically correspond to? [10 pts]

(c) Plot your model on a temperature-density ($T - \rho$) plot, as in Figure 8.1 (you don't need to plot the boundary lines for different equations of state). Then overplot one of your MESA solar mass models, perhaps the one from Computer Problem 0 (the one found at the end of the MESA notes on the course webpage). If you didn't save the density or temperature interior profile, it's quick to rerun MESA agagin with those quantities included. [5 pts]

(d) Derive those first 4 terms of the expansion in Equation (10.37). Hint: First show or explain that if $\theta(\xi)$ is a solution of the Lane-Emden equation, then so must $\theta(-\xi)$. This motivates you to try a solution of the form below of even powers which you can plug into the Lane-Emden equation:

$$\theta(\xi) = C_0 + C_2\xi^2 + C_4\xi^4 + C_6\xi^6 + \dots \tag{10.38}$$

Don't forget to use any boundary conditions you may need. [10 pts]

n	Name	ξ_1	$\rho_c/\overline{\rho}$	N_n	W_n	Θ_n	ρ_c [g cm^{-3}]	P_c [dyne cm^{-2}]	T_c [K]
0.0	–								
0.5	Hannah								
1.0	–								
1.5	Harrison								
2.0	Rogelio								
2.25	Annie								
2.5	Ali								
2.75	Audrey								
3.0	Mark								
3.25	Alexander								
3.5	Kelly								
3.75	Farhan								
4.0	Bryson								
4.25	Matt								
4.5	Oana								
4.75	Manny								
5.0	–								

Answer:

(a) As suggested, let $x=\xi$, $y=\theta$, and $z=\mathrm{d}\theta/\mathrm{d}x=y'$. Then we have $z'=\mathrm{d}z/\mathrm{d}x=-y^n-2z/x$. It might be easier to see this as the two first-order differential equations:

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1^n - \frac{2}{x}\,y_2. \end{aligned}$$

With the appropriate boundary conditions $y_1=1$ and $y_2=0$ at $x=0$.

The correct values are given in Table 10.1. We use $\mu=0.61$. It is interesting to look at some of these things. For $n=0$ we see the solution is a constant density one, which makes sense.

(b) A plot of all $\theta_n(\xi)$ is given in Fig. 10.1. From the derivation of the polytrope, it is easily seen that the quantities $\theta_n(\xi)$, $\theta_n^n(\xi)$, $\theta_n^{n+1}(\xi)$, and $q(\xi)$ physically correspond to T/T_c, ρ/ρ_c, P/P_c, and m/M respectively.

(c) First use the main boundary condition. We already know that $\theta(\xi=0)=1$ so that $C_0=1$. Plugging in the expansion into Equation (10.12) and going up to C_6 gives

$$6C_2+20C_4\xi^2+42C_6\xi^4=-(C_0+C_2\xi^2+C_4\xi^4+C_6\xi^6)^n.$$

Now we have to match powers in the expansion using the multinomial theorem on the RHS. It helps that we only want terms that go up to the 4th power of ξ. There are only 4 terms in the expansion where that is possible. If we call the 4 terms on the RHS above a, b, c, and d, and note that the sum of their powers must equal n, those 4 terms and their coefficients are

$$\begin{aligned} a^n &\quad ; \quad 1 \\ a^{n-1}b &\quad ; \quad n \\ a^{n-1}c &\quad ; \quad n \\ a^{n-2}b^2 &\quad ; \quad \frac{n(n-1)}{2} \end{aligned}$$

Writing LHS and RHS all out gives

$$6C_2+20C_4\xi^2+42C_6\xi^4=-\left(C_0^n+nC_0^{n-1}C_2\xi^2+nC_0^{n-1}C_4\xi^4+\frac{n(n-1)}{2}C_0^{n-2}C_2^2\xi^4\right).$$

Now the coefficients can be matched. To zeroth order, since $C_0=1$, then $C_2=-1/6$. To second order, we see $C_4=n/120$. To fourth order we have

$$42C_6=-\frac{n^2}{120}-\frac{n(n-1)}{72}.$$

Playing with the common denominator gives $C_6=-n(8n-5)/15120$, thus recovering Equation (10.37).

10.4 Usefulness of polytropes

- First note that unless K is fixed by a known equation of state (like it is for degenerate gases and a few other cases), then one needs to provide a value of n, M, and R to get physical values for the interior properties.

- A degenerate gas is a case where K is fixed by the equation of state. Recall the non-relativistic case in Equation (7.10):

$$P_\mathrm{e}=1.0036\times10^{13}\left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3}\ \mathrm{dyne\,cm^{-2}}.$$

- The exponent on density implies $n=3/2$.

Table 10.1: Solutions for different polytropes.

n	ξ_1	$\frac{\rho_c}{\bar{\rho}}$	N_n	W_n	Θ_n	ρ_c	P_c	T_c
0.0	2.449	1.000	...	0.119	0.500	1.409	1.344e15	7.000e6
0.5	2.753	1.835	2.525	0.212	0.484	2.585	2.388e15	6.778e6
1.0	3.142	3.290	6.283	0.393	0.500	4.635	4.420e15	6.997e6
1.5	3.654	5.990	0.424	0.770	0.538	8.440	8.667e15	7.535e6
2.0	4.353	11.400	0.365	1.638	0.602	16.063	1.843e16	8.421e6
2.5	5.355	23.403	0.352	3.908	0.700	32.977	4.399e16	9.789e6
2.75	6.036	34.950	0.355	6.404	0.768	49.247	7.209e16	1.074e7
3.0	6.897	54.174	0.364	11.048	0.854	76.336	1.244e17	1.195e7
3.25	8.019	88.139	0.379	20.361	0.968	124.19	2.292e17	1.354e7
3.5	9.536	152.87	0.401	40.905	1.121	215.41	4.604e17	1.569e7
4.0	14.972	622.400	0.477	247.56	1.666	877.02	2.787e18	2.331e7
4.25	20.529	1635.8	0.555	866.21	2.218	2305	9.75e18	3.104e7
4.5	31.836	6189.2	0.658	4921.6	3.331	8721.1	5.540e19	4.661e7
4.75	66.387	56562	0.904	90417	6.696	79701	1.018e21	9.370e7
5.0	∞	∞	∞	∞	∞	∞	∞	∞

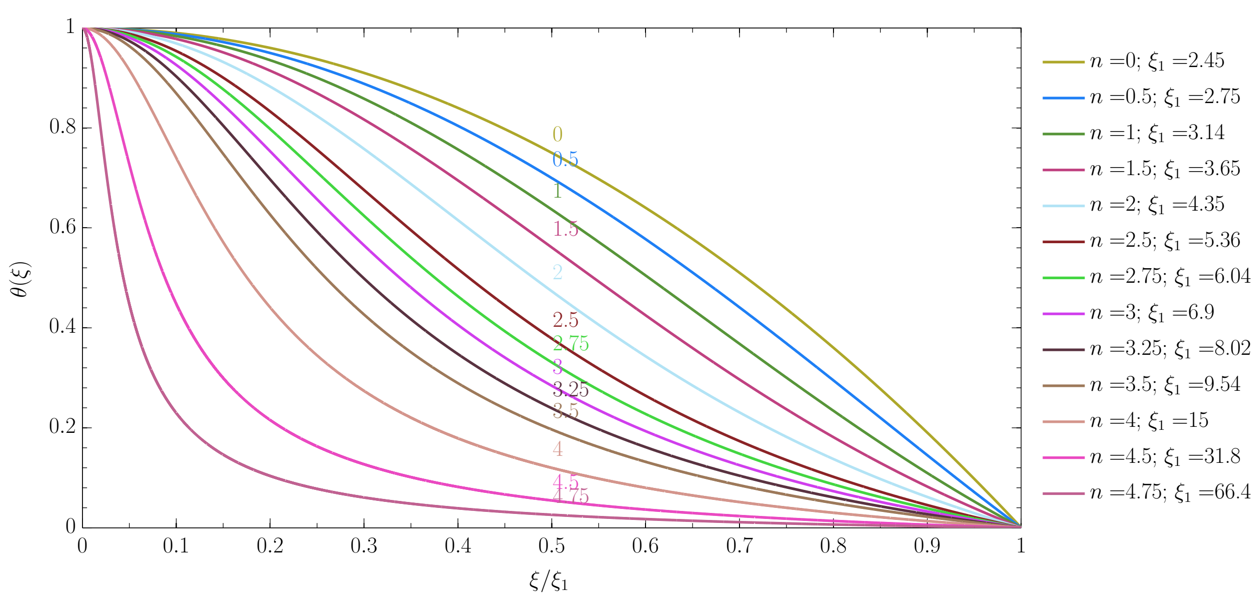

Figure 10.1: Polytropic variable θ_n as a function of ξ/ξ_1 for given values of n.

- Using the expression for K in Equation (10.22) along with the necessary computed values gives

$$K = 2.477 \times 10^{14} \left(\frac{M}{M_\odot}\right)^{1/3} \frac{R}{R_\odot}. \tag{10.39}$$

- This yields a mass-radius relationship

$$\frac{M}{M_\odot} \approx 2.08 \times 10^{-6} \left(\frac{2}{\mu_\mathrm{e}}\right)^5 \left(\frac{R}{R_\odot}\right)^{-3}. \tag{10.40}$$

  Note how this implies that adding mass decreases the star's radius.

- White dwarfs are measured to have masses around $0.6\,M_\odot$. For completely ionized gas, $\mu_\mathrm{e} = 2$, and this gives a radius of about $R \approx 0.015\,R_\odot$, which is about the radius of the Earth.

- In deriving Equation (7.10) we could have substituted the mass of the neutron instead of the electron. Neutrons can become degenerate in special cases too. Then taking $\mu_\mathrm{e} = 1$, the *neutron star* equivalent of Equation (10.40) is

$$\frac{M}{M_\odot} \approx 5 \times 10^{-15} \left(\frac{R}{R_\odot}\right)^{-3}. \tag{10.41}$$

- For a $1\,M_\odot$ star, we find $R \approx 10\,\mathrm{km}$.

- For relativistic electrons, we had from Equation (7.13)

$$P_\mathrm{e} = 1.243 \times 10^{15} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3} \;\mathrm{dyne\,cm^{-2}}.$$

- This now implies a polytropic index $n = 3$. Using the proper values for this case gives

$$K = 3.841 \times 10^{14} \left(\frac{M}{M_\odot}\right)^{2/3}, \tag{10.42}$$

  or, the relation (independent of radius)

$$\frac{M}{M_\odot} = 1.456 \left(\frac{2}{\mu_\mathrm{e}}\right)^2. \tag{10.43}$$

- This is the Chandrasekhar limiting mass for a degenerate, relativistic electron gas. We'll come back to this again discussing late-stage evolution.

- Another interesting insight is for radiation pressure.

- Consider the case for an ideal gas where radiation pressure is also contributing. Recall Equation (12.1)

$$P = \frac{R_g}{\mu}\rho T + \frac{1}{3}aT^4 = \frac{R_g}{\mu\beta}\rho T,$$

  where $\beta = P_\mathrm{gas}/P$ is assumed constant throughout the star.

- If we write this as

$$P_\mathrm{rad} = \frac{1-\beta}{\beta}P_\mathrm{gas} = \frac{1-\beta}{\beta}\frac{\rho R_g}{\mu}T = \frac{1}{3}aT^4, \tag{10.44}$$

  then we can solve for T, which gives

$$T = \left(\frac{3R_g}{a\mu}\frac{1-\beta}{\beta}\right)^{1/3} \rho^{1/3}. \tag{10.45}$$

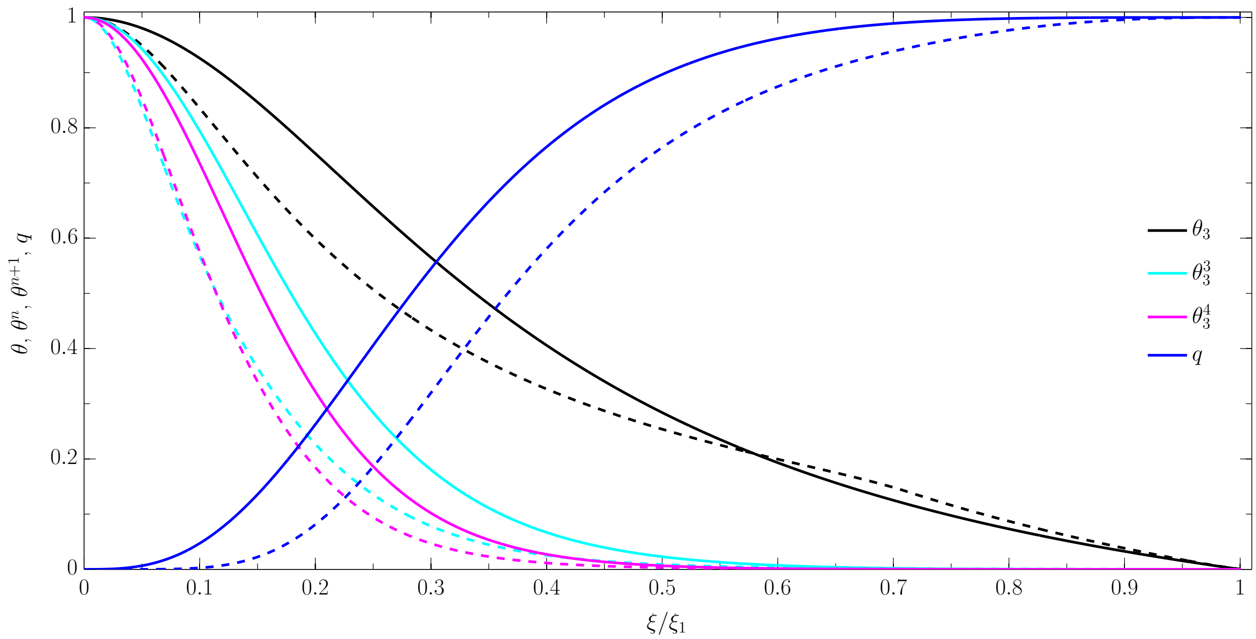

Figure 10.2: Various quantities for a $n = 3$ polytrope compared to Model S. For each colored curve the solid line is the polytropic solution, and the dashed line is the model Sun.

- Using this to plug back into the total pressure P, we can find

$$P = \left(\frac{3R_g^4}{a\mu^4}\right)^{1/3}\left(\frac{1-\beta}{\beta^4}\right)^{1/3}\rho^{4/3} = K\rho^{4/3}, \tag{10.46}$$

  which demonstrates this is an $n = 3$ polytrope.

- Here, K is again a free parameter given some value of β. But we also have another expression for K from Eq. (10.22).

- Equating those 2 Ks and using the right values for $n = 3$ gives

$$\frac{1-\beta}{\beta^4} = 0.002996\mu^4\left(\frac{M}{M_\odot}\right)^2. \tag{10.47}$$

- This shows, as expected, that more massive stars have larger contributions from radiation pressure.

- For $\mu^2 M/M_\odot = 1$, $\beta = 0.997$. But for $\mu^2 M/M_\odot = 50$, $\beta \approx 0.5$.

- One can also look at the temperature using the above result to find

$$T = 4.62 \times 10^6\,\beta\mu\left(\frac{M}{M_\odot}\right)^{2/3}\rho^{1/3}. \tag{10.48}$$

- Finally, consider the Sun.

- It is apparent from the polytrope solutions (Table 10.1) that Θ_n does not change much with n, so our approximation for the central temperature found in Problem 9.1 is quite good, where we effectively found that $\Theta = 1$. The numerical central temperature is $T_c = 1.57 \times 10^7\,\mathrm{K}$.

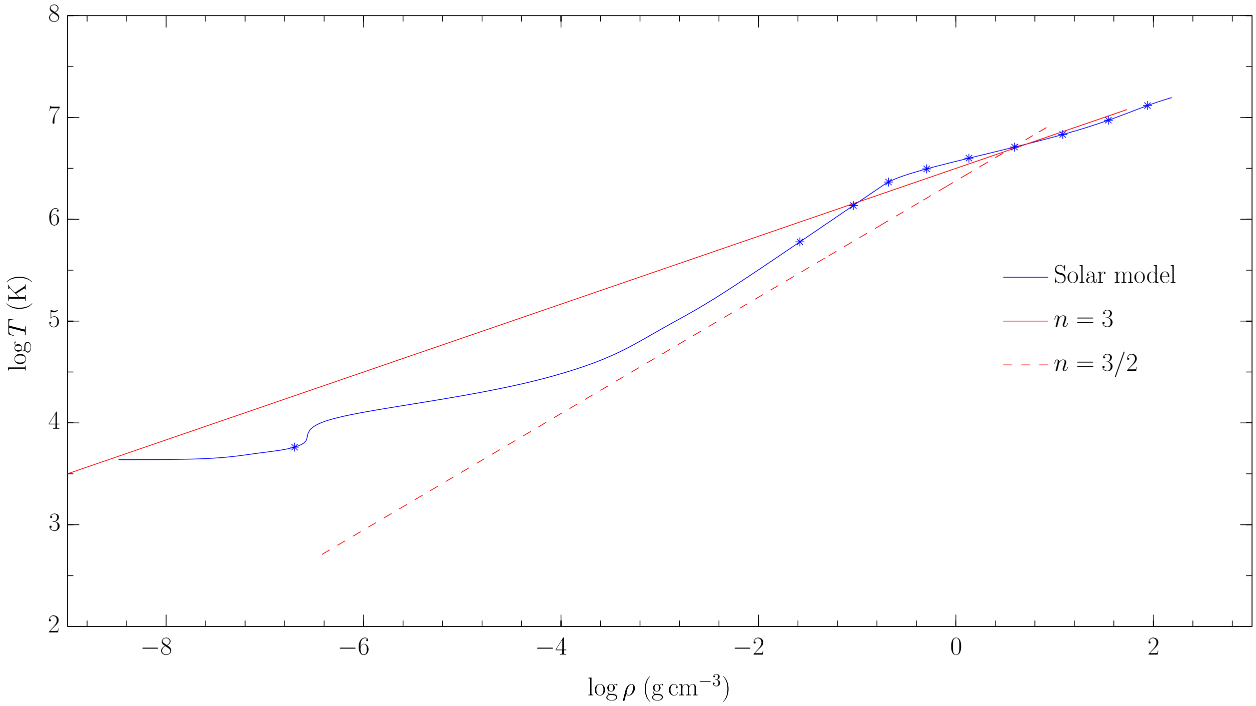

Figure 10.3: A modern solar model compared to 2 different polytropic models. The blue dots on the solar model denote the interior radial coordinate from $r = (0.1 - 1.0)R_\odot$ in steps of 0.1. This particular solar model goes slightly beyond the photosphere.

- We see that other solutions however vary quite widely for different indices. For example, The W_n can be directly compared with our prior estimates of central pressure in Section 9.1. For the Sun we found that the constant is about $261/4\pi \approx 21$, with total numerical value $P_c = 2.34 \times 10^{17}$ dyne cm². This is the first piece of evidence that the Sun is well described by a $n = 3$ polytrope.

- We also know the Sun's central density is about $\rho_c = 153\,\mathrm{g\,cm^{-3}}$, and ratio with mean density $153/1.4 \approx 110$.

- All of these comparisons point to the Sun being somewhere between $n = 3.0 - 3.5$, at least in the central regions.

- In Figure 10.2 we plot a $n = 3$ polytrope and compare with a modern solar model (Model S). What is remarkable is how well the polytrope approximates the Sun without any knowledge of energy generation or energy transfer or chemical composition variations.

- Figure 10.3 shows a detailed solar model in the $T - \rho$ landscape, compared to an $n = 3$ and $n = 3/2$ polytrope. Near the core, the polytrope is in good agreement with the Sun since the density is quite large. Recall an $n = 3$ polytrope corresponds to $\gamma = 1 + 1/3 = 4/3$, which is the degenerate electron gas case.

- However in the outer layers, the Sun behaves more like an ideal gas because of convection, so a $\gamma = 5/3$, $n = 1/(\gamma - 1) = 1.5$ polytrope is a better approximation.

- Note that the layers from $r = 0.7$ outward constitutes 30% of the star in radius, but only about 0.5% in mass.

Unit 11

Thermodynamics of an Ideal Gas

11.1 First law of thermodynamics

- Here we consider quasistatic changes to the state of a nondegenerate gas to understand its thermodynamic properties. Thermodynamics are "responses" of a gas to perturbations.
- As already stated, the internal energy per unit volume of an ideal gas is

$$u = \frac{3}{2} n k_{\mathrm{B}} T = \frac{3}{2} \frac{\rho k_{\mathrm{B}} T}{\mu m_p} = \frac{3}{2} P. \tag{11.1}$$

- Therefore the average energy per particle is $3/2 k_{\mathrm{B}} T$. Example problem 6.1 arrived at this in a slightly different way.
- We define the specific volume V as the volume corresponding to unit mass, $V = \text{volume/mass} = 1/\rho$. The specific internal energy is the internal energy per unit mass $U = u/\rho$:

$$U = \frac{3}{2} \frac{k_{\mathrm{B}} T}{\mu m_u}. \tag{11.2}$$

- Remember that the first law of thermodynamics tells us that we can (slowly) change the internal energy of gas by adding heat or doing work:

$$\mathrm{d}U = \mathrm{d}Q + \mathrm{d}W, \tag{11.3}$$

where U is the specific internal energy of the matter, V is the specific volume it occupies, and $\mathrm{d}Q$ is some amount of heat added to it.

- The work done is to contract or expand it, so $\mathrm{d}W = -P\mathrm{d}V$.
- The more proper form for our use is

$$\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V. \tag{11.4}$$

This heat partly changes the internal energy of the matter and also potentially changes the volume.

- Keeping the volume constant the first law of thermodynamics becomes

$$c_V = \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_V = \frac{\mathrm{d}U}{\mathrm{d}T} = \frac{3}{2} \frac{k_B}{\mu m_u} = \frac{3}{2} \frac{R}{\mu}. \tag{11.5}$$

This is the specific heat at constant volume.

- Now consider how to arrive at an expression for constant pressure. Rewrite the first law

$$\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V + V\mathrm{d}P - V\mathrm{d}P = \mathrm{d}U - V\mathrm{d}P + \mathrm{d}(PV), \tag{11.6}$$

$$\mathrm{d}(PV) = \mathrm{d}(Nk_{\mathrm{B}}T) = \frac{k_{\mathrm{B}}}{\mu m_u}\mathrm{d}T, \tag{11.7}$$

remembering that $N = nV = n/\rho = \rho/(\mu m_u \rho) = 1/\mu m_u$.

- Then

$$\mathrm{d}Q = \mathrm{d}U - V\mathrm{d}P + \frac{k_{\mathrm{B}}}{\mu m_p}\mathrm{d}T = \frac{5}{2}\frac{k_{\mathrm{B}}}{\mu m_u}\mathrm{d}T - V\mathrm{d}P, \tag{11.8}$$

by using Equation (11.2).

- Therefore the specific heat at constant pressure is

$$c_P = \frac{5}{2}\frac{k_{\mathrm{B}}}{\mu m_u} = \frac{5}{2}\frac{R}{\mu}.. \tag{11.9}$$

Note that $c_P - c_V = R/\mu$.

- Note also the ratio of specific heats, $\gamma = c_P/c_V$, which for an ideal gas $\gamma = 5/3$ since the specific heats are constants.

11.2 Adiabatic process

- An **adiabatic process** is one in which no heat is added to the gas ($\mathrm{d}Q = 0$).
- In this special case, we can find expressions relating changes in P and V. Using the above expressions we can show

$$c_V\left(\frac{\mathrm{d}P}{P} + \frac{\mathrm{d}V}{V}\right) = -\frac{k_{\mathrm{B}}}{\mu m_p}\frac{\mathrm{d}V}{V} = (c_V - c_P)\frac{\mathrm{d}V}{V}. \tag{11.10}$$

- Finally we see that

$$\frac{\mathrm{d}P}{P} = -\gamma\frac{\mathrm{d}V}{V} = \gamma\frac{\mathrm{d}\rho}{\rho} = -\frac{\gamma}{1-\gamma}\frac{\mathrm{d}T}{T}. \tag{11.11}$$

- Since γ is constant in this case, such equations can be readily integrated to yield relations such as $PV^{\gamma} = \mathrm{const}$.
- Using the ideal gas law we can also write (just in terms of P, T, and V):

$$\left(\frac{\partial \ln P}{\partial \ln V}\right)_s = -\gamma \equiv -\Gamma_1 \tag{11.12}$$

$$\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_s = \gamma \equiv \Gamma_1 \tag{11.13}$$

$$\left(\frac{\partial \ln P}{\partial \ln T}\right)_s = \frac{\gamma}{\gamma - 1} \equiv \frac{\Gamma_2}{\Gamma_2 - 1} \tag{11.14}$$

$$\left(\frac{\partial \ln T}{\partial \ln V}\right)_s = 1 - \gamma \equiv 1 - \Gamma_3. \tag{11.15}$$

$$\left(\frac{\partial \ln T}{\partial \ln \rho}\right)_s = \gamma - 1 \equiv \Gamma_3 - 1. \tag{11.16}$$

- The s means adiabatic, or at constant entropy, where $\mathrm{d}Q = T\mathrm{d}S$.
- The connection between Eq. (11.12) and (11.13) is clear from the specific volume definition. Similarly for Eq. (11.15) and (11.16).

EXAMPLE PROBLEM 11.1: Derive the 2 equations (11.10) and (11.11).

Answer: Begin with Equation (11.7) rewritten here:

$$P\mathrm{d}V + V\mathrm{d}P = \frac{k_\mathrm{B}}{\mu m_p}\mathrm{d}T.$$

For adiabatic processes, the first law gives us that

$$c_V\mathrm{d}T = -P\mathrm{d}V,$$

so that we have

$$P\mathrm{d}V + V\mathrm{d}P = -\frac{k_\mathrm{B}}{\mu m_p}\frac{1}{c_V}P\mathrm{d}V.$$

Now divide through by PV and recall the relationship between the 2 specific heats to find

$$c_V\left(\frac{\mathrm{d}V}{V} + \frac{\mathrm{d}P}{P}\right) = (c_V - c_P)\frac{\mathrm{d}V}{V}.$$

Collecting terms gives

$$\frac{\mathrm{d}P}{P} = -\gamma\frac{\mathrm{d}V}{V} = \gamma\frac{\mathrm{d}\rho}{\rho}.$$

The term with the log is just a different way of rewriting the above equation, such as

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln\rho} = \gamma.$$

To get the term with P and T, remember that with the ideal gas law, $P \sim \rho T$ (forget the constants n or k; they'll drop out in the end). So using the product rule,

$$\mathrm{d}P = T\mathrm{d}\rho + \rho\mathrm{d}T,$$

and then divide by P, multiply by γ, and use the ideal gas law. We find

$$\gamma\frac{\mathrm{d}\rho}{\rho} = \gamma\left(\frac{\mathrm{d}P}{P} - \frac{\mathrm{d}T}{T}\right),$$

and collecting terms can then give you Equation (11.14). Using the ideal gas law again with T and ρ will give you Equation (11.15).

Unit 12

Thermodynamics with Photons

12.1 Mixture of ideal gas and radiation: pressure effects

- It should be noted that we have been considering an ideal gas made up of particles only. When radiation is present along with the gas in thermodynamic equilibrium, the photons can cause two changes: (1) a radiation pressure; and (2) ionization effects (see Sec. 12.2). In this case, the adiabatic exponents in Eqs. (11.12-11.15) are not constant anymore, nor are they all equal.

- Considering this mixture, the total pressure is

$$P = P_{\rm gas} + P_{\rm rad} = \frac{\rho k_{\rm B} T}{\mu m_{\rm u}} + \frac{1}{3} a T^4, \tag{12.1}$$

  and specific internal energy density

$$U = \frac{3}{2}\frac{k_{\rm B} T}{\mu m_{\rm u}} + a T^4 V. \tag{12.2}$$

- Since the specific energy depends on volume and temperature $U(T, V)$, quasistatic changes to it in the first law of thermodynamics yields

$$\mathrm{d}Q = \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial U}{\partial V}\right)_T \mathrm{d}V + P\mathrm{d}V. \tag{12.3}$$

- Using the expression for the specific energy and pressure then gives

$$\mathrm{d}Q = \left(4aT^3 V + \frac{3}{2}\frac{k_{\rm B}}{\mu m_{\rm u}}\right)\mathrm{d}T + \left(\frac{4}{3} a T^4 + \frac{k_{\rm B} T}{\mu m_{\rm u} V}\right)\mathrm{d}V. \tag{12.4}$$

- For an adiabatic process, this equation gives the thermodynamic response to changes in temperature and volume. It can then be nicely rewritten

$$\left(12 P_{\rm rad} + \frac{3}{2} P_{\rm gas}\right)\frac{\mathrm{d}T}{T} + (4P_{\rm rad} + P_{\rm gas})\frac{\mathrm{d}V}{V} = 0. \tag{12.5}$$

- To evaluate expressions as in Equations (11.12)-(11.15), it's also useful to have a pressure differential term. Using Equation (12.1) we can write

$$\begin{aligned} \mathrm{d}P &= \left(\frac{4}{3} a T^4 + \frac{RT}{\mu V}\right)\frac{\mathrm{d}T}{T} - \frac{R}{\mu}\frac{T}{V}\frac{\mathrm{d}V}{V}, \\ &= (4P_{\rm rad} + P_{\rm gas})\frac{\mathrm{d}T}{T} - P_{\rm gas}\frac{\mathrm{d}V}{V}. \end{aligned} \tag{12.6}$$

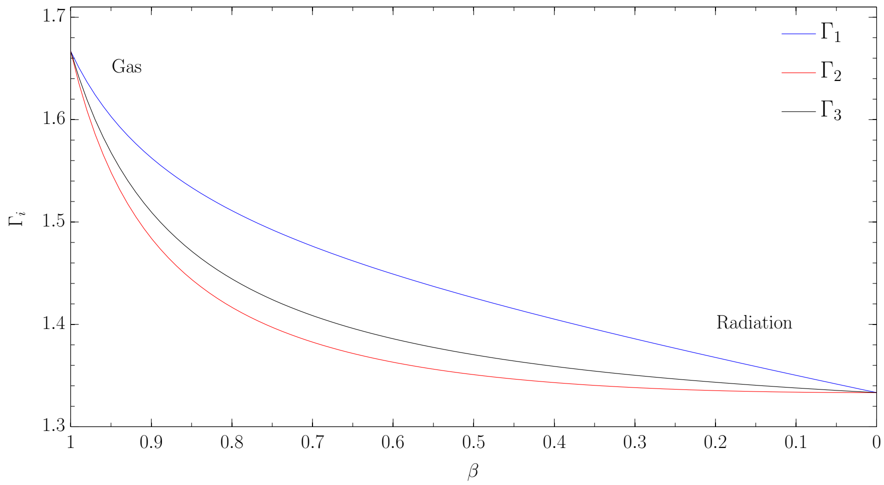

Figure 12.1: The various adiabatic exponents for different mixtures of ideal gas particles and photons.

- Plugging this into Equation (11.12) gives

$$(4P_{\rm rad} + P_{\rm gas})\frac{\mathrm{d}T}{T} + [\Gamma_1(P_{\rm rad} + P_{\rm gas}) - P_{\rm gas}]\frac{\mathrm{d}V}{V} = 0. \tag{12.7}$$

- Comparing Equation (12.5) and Equation (12.7) allows us to solve for Γ_1. It simplifies things to consider the fractional gas pressure as was done previously

$$\beta \equiv \frac{P_{\rm gas}}{P_{\rm gas} + P_{\rm rad}}. \tag{12.8}$$

- You can then show that

$$\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}. \tag{12.9}$$

- For a gas of particles, $\beta = 1$, and therefore $\Gamma_1 = 5/3 = \gamma$, which is what we already found for an ideal gas. For a photon gas, $\beta = 0$ and $\Gamma_1 = 4/3$.

- In a similar fashion,

$$\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}, \tag{12.10}$$

$$\Gamma_3 = \frac{32 - 27\beta}{24 - 21\beta}. \tag{12.11}$$

  See Figure 12.1 for the dependence of these on β.

- Using the equations we just developed, the specific heats can also be computed:

$$c_V = \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_V = c_V^0\,\frac{8 - 7\beta}{\beta}, \tag{12.12}$$

$$c_P = \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_P = c_V^0\,\frac{32/3 - 8\beta - \beta^2}{\beta^2}, \tag{12.13}$$

(12.14)

  where $c_V^0 = \frac{3}{2}\frac{R}{\mu}$ is the ideal gas-only value.

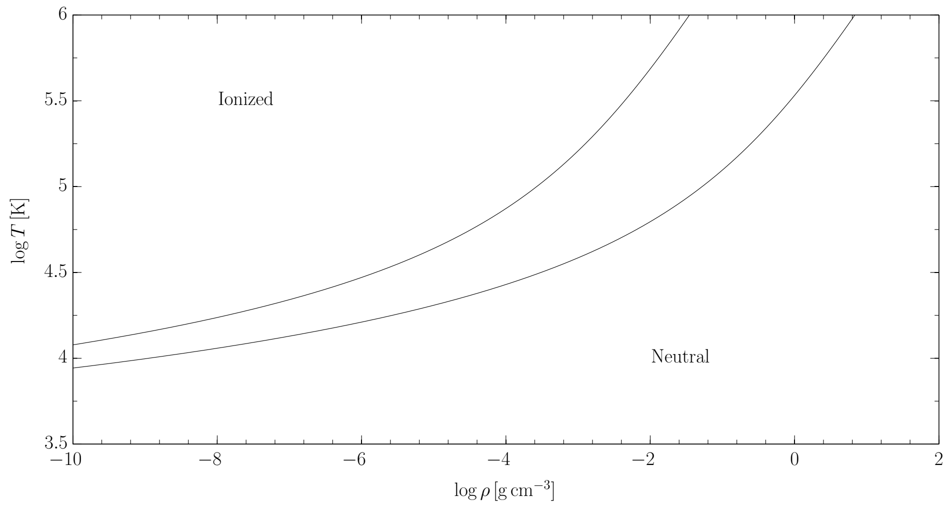

Figure 12.2: Ionization of a pure H gas using Equation (12.19). The lower line represents the state at 50% ionization, while the upper is for 99%.

- Note that the ratio of specific heats gives
$$\frac{c_P}{c_V} = \frac{\Gamma_1}{\beta}, \tag{12.15}$$
which makes sense in the appropriate limits, reducing to what we found before.
- The same procedure can be carried out for mixtures of some degenerate gas too.

12.2 Mixture of ideal gas and radiation: ionization effects

- As mentioned at the beginning of this section, the other consideration is the ionization of the gas that radiation produces, which has profound effects on the thermodynamic state of the gas.
- Let's just consider a hydrogen gas for simplicity in what follows.
- In general radiation causes ionization and subsequent recombination:
$$\mathrm{H}^+ + \mathrm{e}^- \longleftrightarrow \mathrm{H}^0 + \chi_\mathrm{H}, \tag{12.16}$$
where $\chi_\mathrm{H} = 13.6\,\mathrm{eV}$ is the energy needed to ionize hydrogen. We will only consider the ground state.
- To measure the number densities of electrons (n_e), ions (n^+), and neutral H (n^0) in thermodynamic equilibrium, we employ the tools we used in Sec. 6.1 using a Boltzmann distribution.
- After taking into account the appropriate degeneracy factors and energy levels and chemical potentials, we can form the ratio n^+n_e/n^0 using Equation (6.12) and obtain the *Saha equation* for a pure hydrogen gas
$$\frac{n^+n_e}{n^0} = \left(\frac{2\pi m_e k_\mathrm{B}}{h^2}\right)^{3/2} T^{3/2}\mathrm{e}^{-\chi_\mathrm{H}/kT}. \tag{12.17}$$
- We constrain the system to have charge neutrality, $n_e = n^+$ and nucleon number density $n^+ + n^0 = n$. Then we define the fraction of ionization
$$y = \frac{n_e}{n} = \frac{n^+}{n}, \tag{12.18}$$
as we did in Equation (5.16).

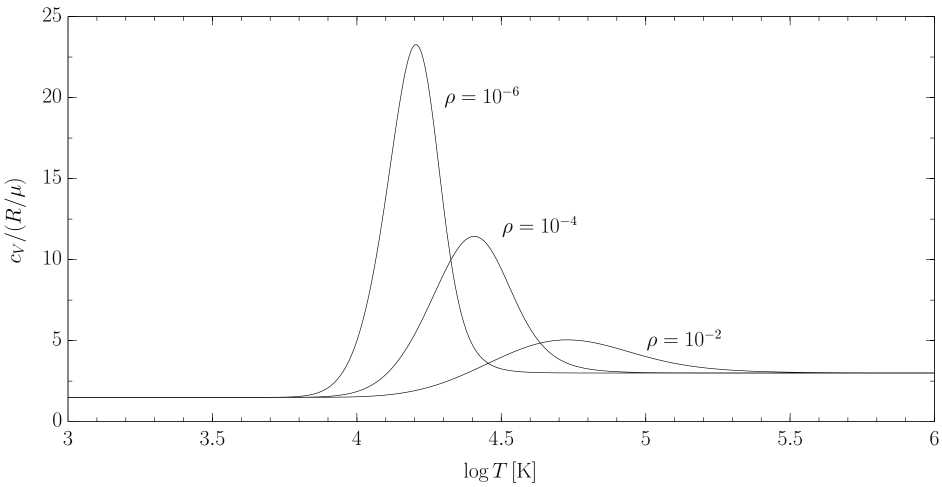

Figure 12.3: The specific heat at constant volume for H at different densities (in $\mathrm{g\,cm^{-3}}$). The familiar value of 3/2 is found for lower temperatures before ionization takes place. Full ionization occurs at the highest temperatures where the value reaches 3, which is twice the value of neutral gas because the number of particles per gram is twice as large.

- Then the Saha equation is
$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e k_\mathrm{B}}{h^2}\right)^{3/2} T^{3/2}\mathrm{e}^{-\chi_\mathrm{H}/kT}. \tag{12.19}$$

- We already see that at high temperatures we expect either collisions or a strong radiation field to ionize the gas $y \to 1$.

- We recognize for a pure hydrogen gas that $n = \rho/\mu m_\mathrm{u}$ with $\mu = 1$, so Equation (12.19) can be solved for a given ionization fraction in terms of temperature and mass density.

- Figure 12.2 shows the necessary conditions for 50% and 99% ionization.

- Note that at about 10^4 K is the half ionization point for hydrogen, only weakly dependent on density.

- A good rule of thumb is that a temperature for half ionization is $\chi/kT \sim 10$ to within a factor of a few depending on density.

- Recall that $1eV \sim 10^4$ K. So , for example, the first ionization potential of He is 24.6 eV. Thus at about 3×10^4 we'd expect ionization to take place, and about double that temperature for removing the 2nd He electron.

- Given the number densities, we can compute the pressure and internal energy as in previous cases and then the full thermodynamic set of quantities.

- Recognize that the pressure
$$P = (n_e + n^+ + n^0)k_\mathrm{B}T = (1+y)N\rho k_\mathrm{B}T, \tag{12.20}$$
since $N\rho = n = n^+ + n^0$. N is the total nucleon number (ions plus neutrals) per unit mass, and is independent of density, thus constant.

- The specific internal energy is
$$U = \frac{3}{2}(1+y)\frac{n}{\rho}k_\mathrm{B}T + y\frac{n}{\rho}\chi_\mathrm{H} = \frac{3}{2}(1+y)Nk_\mathrm{B}T + yN\chi_\mathrm{H}, \tag{12.21}$$
This can be understood since, to completely ionize the gas, we need to add $N\chi_\mathrm{H}$ to strip off the electrons, and another $3/2NkT$ to bring the ions up to the ambient temperature.

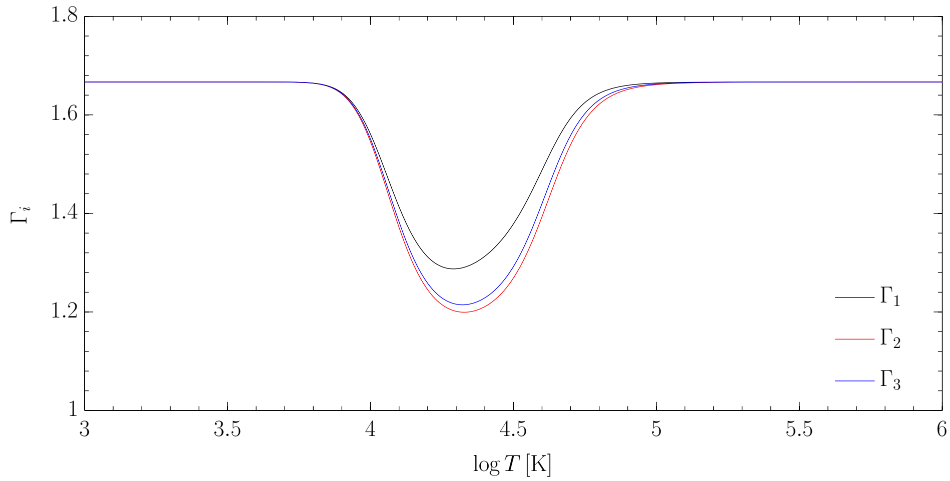

Figure 12.4: Adiabatic exponents for ionized H at a density of $10^{-4}\,\mathrm{g\,cm^{-3}}$.

- From these expressions the specific heats can be computed from the appropriate differentials. Since $U = U(T, y)$, and $c_V = \mathrm{d}U/\mathrm{d}T$, we get extra terms.

- It works out to be

$$c_V = \frac{3}{2} N k_\mathrm{B}(1+y) \left[1 + \frac{2}{3} D(y) \left(\frac{3}{2} + \frac{\chi_\mathrm{H}}{k_\mathrm{B} T} \right)^2 \right], \tag{12.22}$$

  where

$$D(y) = \frac{y(1-y)}{(2-y)(1+y)}. \tag{12.23}$$

- Note that $D(0) = D(1) = 0$, so the specific heat only changes by $1 + y$ as the gas goes from neutral to fully ionized. But D is finite for intermediate values of y, thus bringing in contributions from the other terms.

- A few examples of c_V are shown in Figure 12.3 for several densities.

- Finally, the adiabatic exponents can be computed in similar ways using the prior results and Eqs. (11.12)-(11.15).

- Their values for a density of 10^{-4} across the ionization fraction range is shown in Figure 12.4.

- The Γ_i all take their ideal gas values for full ionization and for complete neutrality.

- Where ionization occurs, the values decrease quickly, and then increase again as ionization completes.

- To understand this, consider adiabatic compression of the gas, and let's focus on Γ_3, which, according to Equation (11.15), relates the temperature and volume.

- Before ionization, we see the value of 5/3 for the neutral gas, which simply means it is heating up under compression as $T \sim \rho^{2/3} \sim V^{-2/3}$.

- When ionization starts to occur, the value decreases, and the temperature sensitivity on volume is weaker. The energy is used to ionize the gas, instead of heating it up as quickly.

12.3 Useful ideal gas equations

Let's collect many of the useful relationships:

$$P = \frac{\rho R T}{\mu} = \frac{\rho k_{\rm B} T}{\overline{m}} \tag{12.24}$$

$$P\rho^{-\gamma} = \text{const} \tag{12.25}$$

$$c_s = \sqrt{\frac{\gamma R T}{M}} = \sqrt{\frac{\gamma P}{\rho}} \tag{12.26}$$

$$C_P - C_V = R/\mu \tag{12.27}$$

$$C_V = \frac{3}{2}\frac{R}{\mu} \tag{12.28}$$

$$C_P = \frac{5}{2}\frac{R}{\mu} \tag{12.29}$$

$$U = \int_0^M C_V T \,\mathrm{d}m' \tag{12.30}$$

Unit 13

Energy Transport: Radiation

Energy liberated in stellar interiors is transferred to the surface by radiation, convection, and conduction. We are not considering here radiation from a stellar photosphere, only the movement of photons in stellar interiors.

13.1 Basics

- The basic idea is that photons emitted in hot regions of a star are absorbed in cooler regions of a star, thus "transferring" energy from hot to cool.
- As we'll soon see, the "efficiency" of this transfer will depend on the temperature gradient. A very rough approximation of the gradient for the Sun is $\mathrm{d}T/\mathrm{d}r \approx (T_{\mathrm{surf}} - T_{\mathrm{core}})/(r_{\mathrm{surf}} - r_{\mathrm{core}}) \approx T_c/R_\odot \approx -2.25 \times 10^{-4}\,\mathrm{K\,cm^{-1}}$.
- Figure 13.1 shows this number compared to the "true" temperature gradient in the interior of the Sun. Clearly there is more physics taking place than we've considered so far.
- The efficiency of radiation will also depend on the ability of the photons to travel freely.
- Consider the luminosity roughly as the (total radiation energy stored in the star) divided by the (escape time for photons).
- The radiation energy is the energy density of photons (Eq. 8.4), say, at the central temperature of the star (the Sun in this case)
$$E_\gamma = aT_c^4 \cdot \frac{4\pi}{3} R_\odot^3. \tag{13.1}$$
- For the photon escape time, let's first consider that the Sun were completely transparent to photons. The time would then be $R_\odot/c = 2.32$ s. The resulting luminosity would be quite large!

EXAMPLE PROBLEM 13.1: If we regard the Sun as a large cavity filled with photons, compute the luminosity by estimating the total energy stored in the radiation field and the Sun becoming completely transparent. Express the luminosity in $L_\odot$.

Answer: The luminosity would be the total energy divided by the escape time, E_γ/τ where $\tau = R_\odot/c$. Then
$$L = aT_c^4 \frac{4}{3}\pi R_\odot^2 c = 2.33 \times 10^{47}\,\mathrm{erg\,s^{-1}} = 6.1 \times 10^{13} L_\odot. \tag{13.2}$$
Clearly something is wrong here.
Instead, consider that a photon travels a distance ℓ before being scattered. In the case of a random walk,

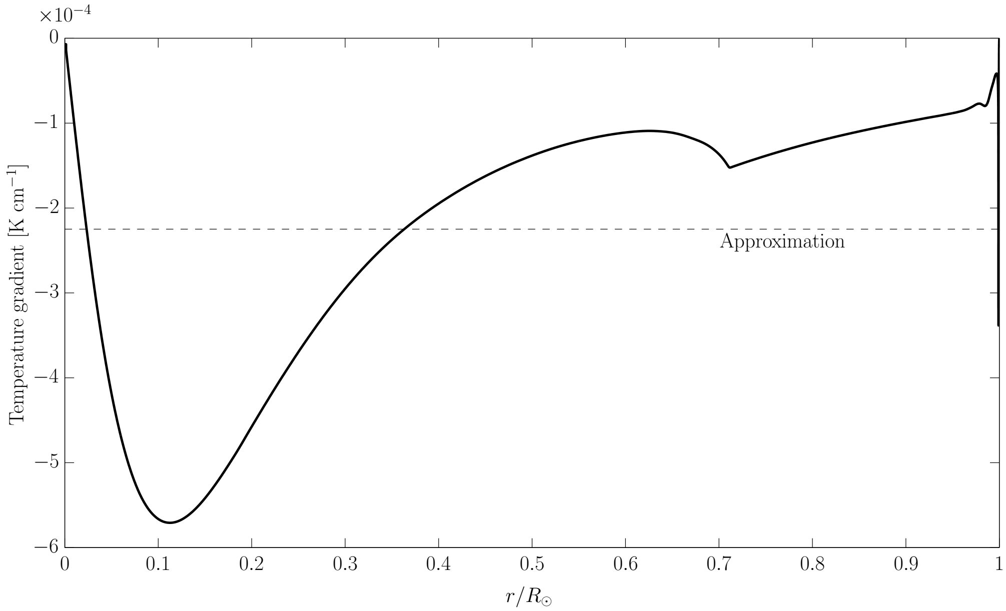

Figure 13.1: The interior temperature gradient of a solar model. Also plotted is a simple estimate of the gradient $\sim T_c/R_\odot$.

the total travel distance of a photon would be on the order of $R_\odot(R_\odot/\ell)$. The escape time is then $R_\odot^2/(c\,\ell)$. Using the luminosity of the Sun to solve for ℓ gives $\ell \approx 10^{-3}$ cm.

- More formally, the mean free path of photons can be expressed as

$$\ell_{\rm ph} = \frac{1}{\kappa\rho}, \tag{13.3}$$

  where κ is some absorption coefficient (in units of cross section per unit mass) that will be given a physical meaning later, and ρ is the mass density.

- Again for some typical interior solar values. $\kappa \approx 1\,{\rm cm}^2\,{\rm g}^{-1}$ and simply a mean density of $\rho \approx 1.4\,{\rm g\,cm}^{-3}$, gives a mean free path of $\ell_{\rm ph} \approx 1$ cm, not too inconsistent with the earlier estimate, but still quite small.

- Nonetheless, radiative transport occurs by the non-vanishing net flux outward, due to the hotter material below which sets up the gradient.

- Because of the small mean free path, transport can be treated as a **diffusion process** in the interior. (Near the surface, however, this simplification starts to break down).

13.2 Diffusion

- Quick and dirty derivation of Fick's Law of diffusion, just to get the point across.

- Consider **particles** diffusing (randomly) in 3D space at some boundary r.

- Let n be the particle number density, $\overline{v}$ be the mean velocity, and ℓ the mean free path, such that $\ell = 1/\sigma n$, with σ the cross section.

- Consider isotropy. Then about 1/3 of the particles will be moving in the $\hat{r}$ direction. About 1/2 of those will be moving in the $-\hat{r}$ direction

- Flux is a quantity (like number of particles or energy) per unit area per unit time.
- From one direction, the particle flux is

$$F_+ = \frac{1}{6} n_{r-\ell} \overline{v}_{r-\ell} \tag{13.4}$$

- From the other direction

$$F_- = \frac{1}{6} n_{r+\ell} \overline{v}_{r+\ell} \tag{13.5}$$

- Net flux

$$F = F_+ - F_- = \frac{1}{6} \overline{v} (n_{r-\ell} - n_{r+\ell}), \tag{13.6}$$

assuming that $v_{r-\ell} \approx v_{r+\ell} = \overline{v}$.

- If the mean free path does not change on the scale of the density gradient, then

$$\begin{aligned} F &= \frac{1}{6} \overline{v} \left[n_{r-\ell} - n_r - (n_{r+\ell} - n_r) \right] \\ &= \frac{1}{6} \overline{v} \left[-\ell \frac{\mathrm{d}n}{\mathrm{d}r} - \ell \frac{\mathrm{d}n}{\mathrm{d}r} \right] \\ F &= -D \, \nabla_r n, \end{aligned}$$

where the diffusion coefficient $D = 1/3 \, \overline{v} \, \ell$. This is Fick's Law. Again, if ℓ is large, this fails.

- This is generic. On the left you have a flux (in this case of number of particles) and on the right a gradient of density (in this case number density of particles). Note that the flux is carried from a high concentration to a low concentration of particles.
- But we want to compute the flux of diffusing radiative energy. So we need an energy density.
- For photons, we can just let $\overline{v} = c$, $\ell = \ell_{\mathrm{ph}} = 1/\kappa\rho$, and $n = u$. See Equation (8.3) and note that $u = 3P$ for a relativistic system, as derived previously, which gives

$$u = aT^4. \tag{13.7}$$

- So then the radiative flux F_{rad} is

$$F_{\mathrm{rad}} = -\frac{4ac}{3} \frac{T^3}{\kappa\rho} \frac{\mathrm{d}T}{\mathrm{d}r}. \tag{13.8}$$

- The local luminosity at any point passing through a sphere of radius r is $L(r) = 4\pi r^2 F_{\mathrm{rad}}$, so then rearranging we have

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac} \frac{\kappa\rho}{r^2} \frac{L}{T^3}. \tag{13.9}$$

- This is a fundamental equation of stellar structure.

13.3 Frequency dependence of radiation

- What we just did was too simple, even in the diffusion approximation. Our answer is in fact integrated over all photon energies.
- In principle, there is a frequency dependence on the flux F_ν since the energy density and the opacity are partitioned in frequency.

- Let us go back to Equation (13.7) and instead consider

$$u_\nu = \frac{4\pi}{c} B_\nu(T), \tag{13.10}$$

where B is the Planck function for a blackbody radiator

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_{\rm B}T} - 1}. \tag{13.11}$$

This is just from our Bose-Einstein distribution function, Equation (8.1), written in terms of frequency instead of momentum.

- Also keep in mind that the integrated Planck function

$$B(T) = \int_0^\infty B_\nu(T)\,{\rm d}\nu = \frac{ac}{4\pi} T^4. \tag{13.12}$$

- Fick's Law now becomes

$$F_\nu = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{{\rm d}B_\nu}{{\rm d}r} = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{{\rm d}B_\nu}{{\rm d}T} \frac{{\rm d}T}{{\rm d}r}. \tag{13.13}$$

- The total flux integrated over all frequencies is then

$$F_{\rm rad} = \int F_\nu\,{\rm d}\nu = -\frac{4\pi}{3} \frac{1}{\rho} \frac{{\rm d}T}{{\rm d}r} \int_0^\infty \frac{1}{\kappa_\nu} \frac{{\rm d}B_\nu}{{\rm d}T}\,{\rm d}\nu. \tag{13.14}$$

- Comparing Equation (13.14) with Equation (13.8), we see that the κ in the latter is

$$\frac{1}{\kappa} = \frac{\pi}{ac} \frac{1}{T^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{{\rm d}B_\nu}{{\rm d}T}\,{\rm d}\nu. \tag{13.15}$$

- But since

$$\int_0^\infty \frac{{\rm d}B_\nu}{{\rm d}T}\,{\rm d}\nu = \frac{\rm d}{{\rm d}T} \int_0^\infty B_\nu {\rm d}\nu = \frac{{\rm d}B}{{\rm d}T} = \frac{ac}{\pi} T^3, \tag{13.16}$$

where $B = acT^4/4\pi$ (the integral over all frequencies), we can then define

$$\frac{1}{\kappa_{\rm R}} \equiv \frac{1}{\kappa} = \left(\int_0^\infty \frac{1}{\kappa_\nu} \frac{{\rm d}B_\nu}{{\rm d}T}\,{\rm d}\nu\right) \left(\int_0^\infty \frac{{\rm d}B_\nu}{{\rm d}T}\,{\rm d}\nu\right)^{-1}, \tag{13.17}$$

where $\kappa_{\rm R}$ is the *Rosseland mean opacity.*

- All this implies is that Equations (13.8) and (13.9) should replace the opacity by the Rosseland mean opacity:

$$F_{\rm rad} = -\frac{4ac}{3} \frac{T^3}{\kappa_{\rm R} \rho} \frac{{\rm d}T}{{\rm d}r}, \tag{13.18}$$

$$\frac{{\rm d}T}{{\rm d}r} = -\frac{3}{16\pi ac} \frac{\kappa_{\rm R} \rho}{r^2} \frac{L}{T^3}. \tag{13.19}$$

- Note that this weighted opacity gives high frequencies more weight than lower ones (as one could find by differentiating).

- Before we go onto using these expressions to understand stellar structure, let's look at a few of the major sources of $\kappa_{\rm R}$.

Unit 14

Opacity sources

14.1 Kramer's Laws

- The opacity κ determines how the flux is transported by radiation, and so is an important quantity.
- Computing opacities is hard stuff, only a few groups in the world have succeeded (LANL, LLNL). These computations require full quantum-mechanical treatments.
- In general, opacities can be approximated as power laws of the form

$$\kappa = \kappa_0 \rho^n T^{-s} \ \ [\mathrm{cm}^2\, \mathrm{g}^{-1}], \tag{14.1}$$

  similar to how we treated energy generation rate and its dependence on these parameters.
- The goal is to compute *total* Rosseland mean opacities from all possible sources, including the following four:

1. Scattering of photons from electrons
   - Photons, or electromagnetic radiation, can cause free electrons to oscillate at the same frequency if the energy is low enough. These electrons then radiate this energy. This is a scattering process.
   - For relativistic free electrons, this is known as Compton scattering, but for conditions inside most stars, this is Thomspon scattering.
   - For temperatures below about 1 billion Kelvin (thermal energies below electron rest mass energy), the cross section for scattering off electrons is frequency independent.
   - One finds for the opacity

   $$\kappa_{\mathrm{e}} = \frac{n_{\mathrm{e}}\sigma_{\mathrm{e}}}{\rho} \qquad \mathrm{cm}^2\, \mathrm{g}^{-1}, \tag{14.2}$$

     where the cross section is given in the Thompson scattering prescription

   $$\sigma_{\mathrm{e}} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.6652 \times 10^{-24}\, \mathrm{cm}^2. \tag{14.3}$$

     (Note the classical electron radius buried in there).
   - Scattering off electrons will only occur in a highly ionized gas where a sufficient number of electrons are present. Looking back at Equation (5.14) and considering a gas with inconsequential metals gives

   $$\kappa_e \simeq 0.2(1 + X). \tag{14.4}$$

- If there are many metals or ionization is incomplete then the electron densities have to be computed more carefully.
- Below 10,000 K hydrogen is not ionized at low pressure and so this opacity does not contribute to the Rosseland mean.
- For this case, the exponents $n = s = 0$.

EXAMPLE PROBLEM 14.1: Derive Equation (14.4) in the full ionization case.

Answer: The electron density from earlier is

$$n_\mathrm{e} = \frac{\rho}{\mu_\mathrm{e} m_\mathrm{u}}.$$

We showed in an example problem that

$$\mu_\mathrm{e} \simeq \frac{2}{1+X}.$$

So we have

$$\kappa_e = \frac{\sigma_e}{\mu_e m_\mathrm{u}} = (1+X)\frac{\sigma_e}{2m_\mathrm{u}} \simeq 0.2\,(1+X),$$

Using the values of the cross section and the amu.

2. Free-free absorption
   - A single free electron cannot absorb a photon while still conserving energy and momentum.
   - If an ion is nearby, this absorption is possible however.
   - In this case, it can be shown that the opacity

$$\kappa_{\mathrm{f-f}} \approx 10^{23} \frac{Z_c^2}{\mu_\mathrm{e}\mu_\mathrm{I}} \rho T^{-3.5}, \tag{14.5}$$

     where Z_c is some average nuclear charge. Typically a quantum-mechanical gaunt factor needs to be included in the coefficient.
   - Opacities of this form that scale as $\kappa \sim \rho T^{-3.5}$, or $n = 1$ and $s = 3.5$, are known as Kramers opacities.
   - The inverse of this process, when an ion changes the momentum of an electron which then emits a photon, is known as Bremsstrahlung emission
   - Note that for reference, in the core of the Sun the product $\rho T^{-3.5} \approx 5 \times 10^{-23}$. So these opacities are of order 1 to 10 or so.
3. Bound-free absorption
   - Here an ion absorbs a photon which frees a bound electron.
   - Causes continuum opacity at wavelengths bluer than the ionizing photon.
   - It can be shown that the opacity

$$\kappa_{\mathrm{b-f}} \approx 10^{25} Z(1+X)\rho T^{-3.5}. \tag{14.6}$$

     This also has a Kramers form.
   - At temperatures below about 10^4K, this should be used with caution since photons are not energetic enough to ionize a gas.
   - This source of opacity can be larger than free-free absorption by a few orders of magnitude.
4. Bound-bound absorption

- This is photon-induced transitions of electrons between bound levels in an atom or an ion.
- At large temperatures, most photons are ionizing so this contribution is smaller to the total opacity (maybe 10%).
- Very difficult computations since there are millions of absorption lines possible. Need oscillator strengths as well as equation of state.

5. H^-

- We just note that in stellar atmospheres (like our Sun) contributions from negative hydrogen and molecules and grains dominate the opacity.
- Low energy photons starting in the infrared can be absorbed in this bound-free transition.
- What is needed for the H^- opacity are free electrons, which at low temperatures might not be completely abundant from ionization of H.
- Other sources of these electrons are the outer ones from metals, so this is sensitive to metallicity.
- An approximate relation in the range of about 3000-6000K and low densities is

$$\kappa_{\mathrm{H}^-} \approx 2.5 \times 10^{-31} \left(\frac{Z}{0.02}\right) \rho^{1/2} T^9 \tag{14.7}$$

14.2 Consequences

- Figure 14.1 shows the contributions from different opacity sources in a stellar interior.
- At low temperatures and partial ionization, b-b and b-f absorption dominates from the bound electrons.
- As ionization occurs at higher temperatures, free-free opacity takes over, but as T continues to increase, $\kappa_{\mathrm{f-f}}$ decreases and scattering from free electrons dominates.
- In reality, all of these processes are occuring at any one time and place, so contributions must be added correctly.
- Note that the sum of the Rosseland mean opacities of each component is not the same as the Rosseland mean of the sum
- Figure 14.2 shows opacity data from the OPAL project.[1] We see that the Sun's opacity can reach 10^5 or so near the surface where atoms recombine.
- Figure 14.2 can be understood in a few limiting cases:
  - For the low density, high temperature case (small R), and for the high temperature regime at any density, all H and He is ionized and the electrons are free particles. Here we expect electron scattering, where Equation (14.4) is dominant. Indeed, $\kappa_e \approx 0.34\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ for an $X = 0.7$ composition, which agrees with the values over most of the temperature range ($\log_{10} -0.34 \approx -0.47$).
  - Below 10,000 K the opacities all converge to small values. Here, all electrons go to the ground state of H and He. Incoming photons can only cause a bound free transition if their wavelengths are shorter than 912 $\AA$ (for H) or 228 $\AA$ (for HeII). In fact, the whole Lyman series (edge) only extends to 1216 $\AA$ (for H). But at this temperature, the blackbody curve is peaked at about 2880 $\AA$, and there just aren't many photons with such high energies available. The stellar region in which this applies is near the surface, but the Sun does not have such conditions as the plots show. Also see Figure 14.3.

[1] http://opalopacity.llnl.gov/

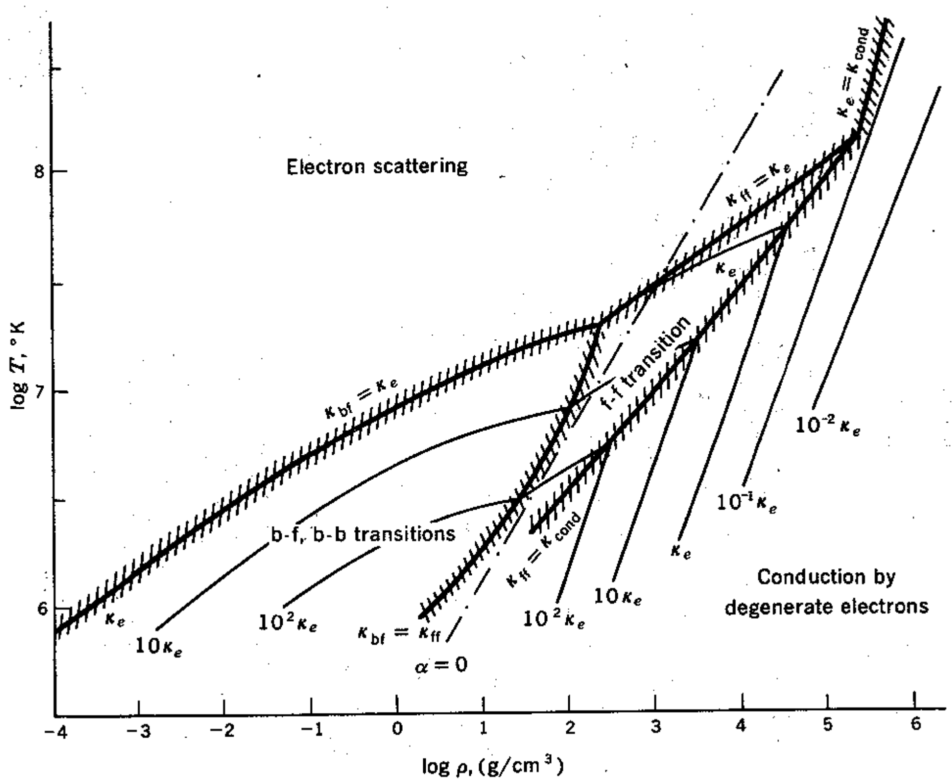

Figure 14.1: Opacity contributions in the $\rho - T$ plane. The values are given in units of electron scattering opacity κ_e. Conduction is not a radiative opacity, and will be discussed in Sec. A.2. From Clayton [1983] after Hayashi et al. [1962].

– At intermediate temperatures we see increased opacity, particularly as the density increases (larger R). Even for the lowest density curve we see a bump around temperatures of 5.0-5.5. Wien's law tells us the wavelengths are in the $100 - 300\,\mathring{A}$ range. This suggests the HeII Lyman edge, even though there aren't many He ions. The bump at about 4.5-4.75 at larger densities could be due to a bound-free absorption of HeI. In any case, these large radiative opacities are found in the H and He ionization zones in stars.

– Note the Kramers shape to the opacities, which decrease at higher temperatures.

– Keep in mind however that these are mean opacities, independent of individual wavelengths.

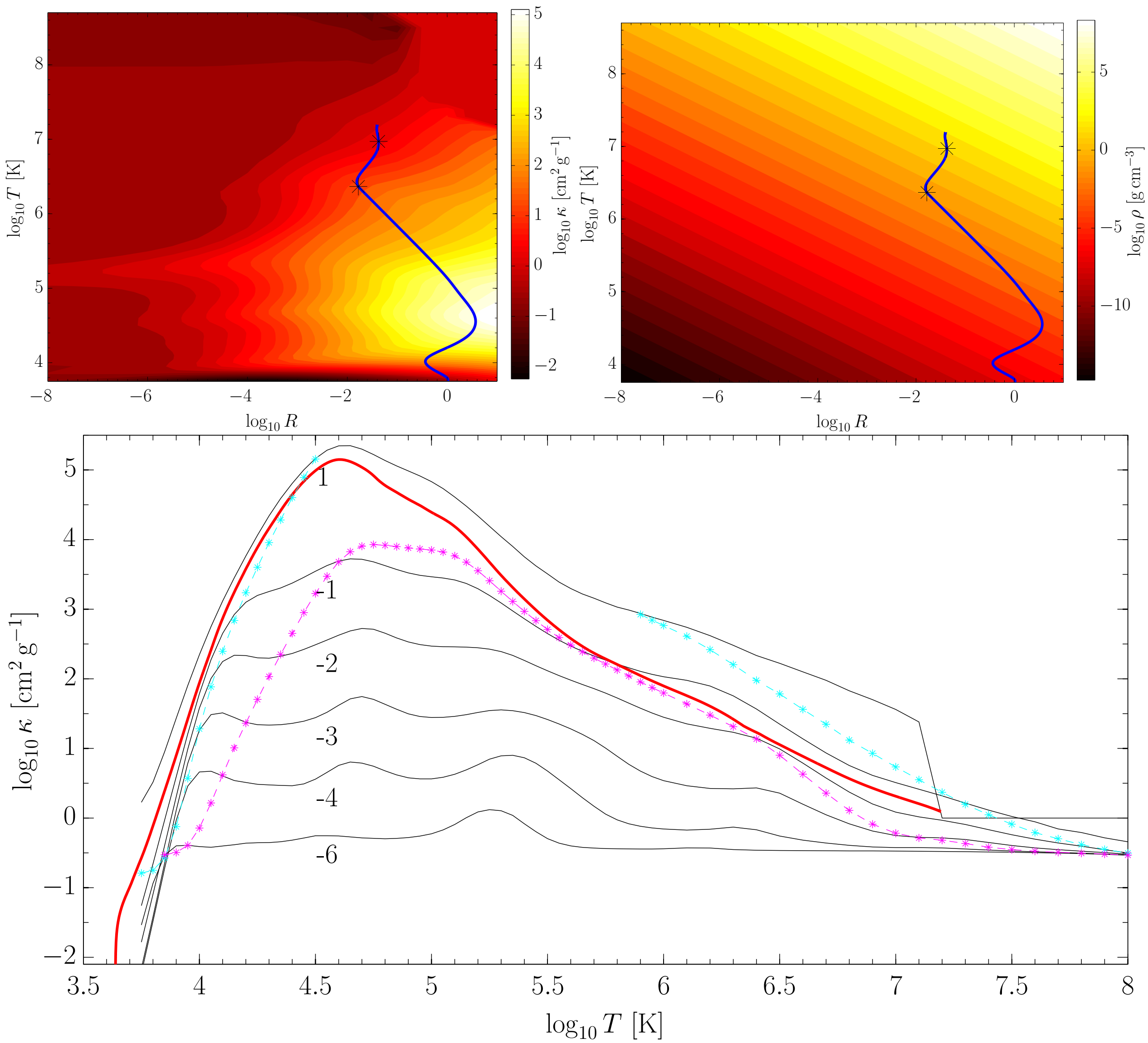

Figure 14.2: (Left top): OPAL Rosseland mean opacity data for a solar composition mixture. The x-axis quantity, R, is defined as $R = \rho T_6^{-3}$, where $T_6 = T \times 10^{-6}$ as usual. Since R can be multivalued for any ρ, T pair, on the right panel is plotted the density, for reference. In each case the blue curve are the values for a standard interior solar model, and the 2 black stars represent the boundaries of the core and convection zone ($0.2R_\odot$ and $0.7R_\odot$, respectively). The direction of increasing T is toward the stellar center. (Bottom): Cuts through the opacity values for specific values of $\log R$, given by the labeled curves. The red curve is the opacity from a current solar model. The cyan and the magenta symbols represent the 0.5 and 0.99 ionization curves for pure hydrogen, respectively, as in Figure 12.2. (Specifically the data are from computation 200503120009, Table 73. The calculations consider about 20 species.)

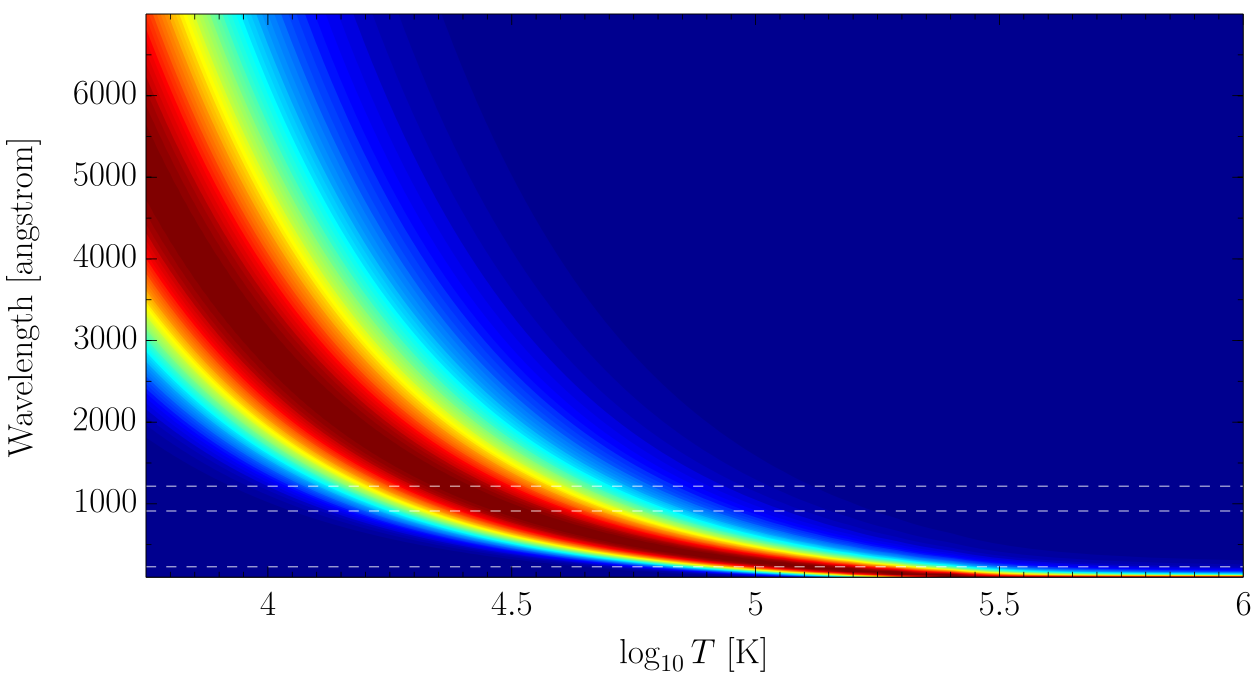

Figure 14.3: Blackbody curves for a range of wavelengths and temperatures. The curve at each temperature is normalized to its maximum value across wavelengths. Wavelength guides from top to bottom are at $1216\,\mathring{A}$, $912\,\mathring{A}$, and $228\,\mathring{A}$.

Unit 15

Convection

Another important carrier of energy from the stellar interior outward is convection.

15.1 Temperature gradients ("dels")

- There are several manipulations we can carry out to make the expressions we derived more useful for later.
- For future use we will need different forms of Equation (13.9). Take hyrdrostatic equilibrium and use logarithmic derivatives:
$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r} = -\frac{Gm\rho}{rP}. \tag{15.1}$$
- Dividing both sides by $\mathrm{d}\ln T/\mathrm{d}\ln r$ gives a new quantity we'll call "del"
$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = -\frac{r^2 P}{Gm\rho}\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{15.2}$$
which is the true driving gradient in the star.
- If we now consider that the luminosity L is carried ONLY by radiation, then we can define "delrad"
$$\nabla_{\mathrm{rad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{P\kappa_{\mathrm{R}}}{T^4}\frac{L}{m}, \tag{15.3}$$
where we used Equation (13.9).
- So if $\nabla = \nabla_{\mathrm{rad}}$, then all the luminosity is radiative. If $\nabla_{\mathrm{rad}} > \nabla$, there is some other transport mechanism of the energy in addition to radiation.
- This quantity is the local slope which is required if all the luminosity were carried by radiation through diffusion.
- In fact, we will use this as a comparison in this unit to a similar quantity we've already introduced in Equation (11.14),
$$\nabla_{\mathrm{ad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{15.4}$$
where this is defined in an "adiabatic" sense, or, i.e., at constant entropy.
- The value of 0.4 comes when considering an ideal gas:
$$\nabla_{\mathrm{ad}} = \frac{\frac{5}{3} - 1}{\frac{5}{3}} = \frac{2}{5} = 0.4. \tag{15.5}$$
This is an important number to keep in mind.

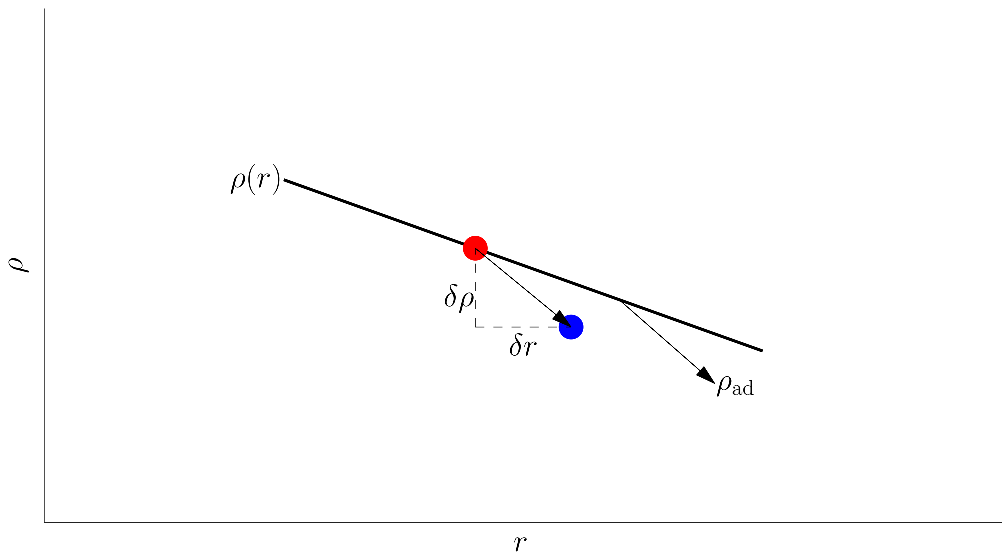

Figure 15.1: Convective instability. The curve $\rho(r)$ denotes the density gradient in some small region of a stellar interior. The arrow is the direction of an adiabat for this material. Take a parcel (red dot) in equilibrium with density ρ, and displace it upwards ($\delta r > 0$) adiabatically. It ends up where the blue dot is. This parcel now has a lower density than the surroundings ($\delta\rho < 0$), and so will continue to rise toward the surface until the conditions change (if they change). The density does not decrease sufficiently fast enough to be stable to convection.

15.2 The convective instability

- Consider in what follows an ideal gas.
- Assume a blob of gas of density ρ and pressure P at point r. It is in equilibrium with its surroundings also then of density ρ and pressure P.
- Let's displace the blob, or perturb it vertically into the medium (at $r + \delta r$) which now has density ρ' and pressure P', which we know are less than the unprimed quantities. What happens to the blob?
- Let ρ^* be the density of the blob. If $\rho^* < \rho'$ then the blob will be buoyant and continue rising: this is unstable. If $\rho^* > \rho'$ then the blob will return to its original position and there is no instablility. So how do ρ^* and ρ' compare?
- Two physically-motivated assumptions: (1) The pressure imbalances are quickly removed by acoustic waves (on the dynamical time scale), so that the pressure of the blob is also P'. (2) Heat is exchanged on the thermal timescale, which is long, so this is an adiabatic displacement.
- We know for an adiabatic displacement that $P/\rho^\gamma = \text{const}$ [Equation (11.11)]. Comparing at bottom and top we can show
$$\rho^* = \rho\left(\frac{P'}{P}\right)^{1/\gamma}. \tag{15.6}$$
- Let's expand the environmental pressure and density about point r to first order:
$$P' = P(r+\delta r) = P(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \ldots \tag{15.7}$$
$$\rho' = \rho(r+\delta r) = \rho(r) + \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r + \ldots \tag{15.8}$$

- Substitute Equations (15.7)-(15.8) into (15.6) and expand (binomial):

$$\rho^* = \rho + \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r. \tag{15.9}$$

- For an instability to occur, $\rho^* - \rho' < 0$, or

$$\rho^* - \rho' = \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r < 0. \tag{15.10}$$

- So, an instability occurs if

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} < \frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{15.11}$$

  where we introduced the adiabatic gradient

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} = \frac{1}{\Gamma}\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{15.12}$$

  where we've denoted $\gamma = \Gamma$ in the adiabatic case.

- This can be interpreted as the density gradient resulting from adiabatic motion in the given pressure gradient.

- Since the gradient of pressure is always negative (hydrostatic equilibrium), instability occurs when the density does not decrease sufficiently rapidly compared to the adiabatic case.

- See Figure 15.1 for a schematic of this.

- Note that it is convention to express Equation (15.11) as

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P} < \frac{1}{\Gamma_1}. \tag{15.13}$$

  (Note since we've divided by a negative number, $\mathrm{d}\ln P/\mathrm{d}r$, the inequality changes). For a fully ionized ideal gas, the RHS is $3/5$.

- Let's now consider the force per unit volume acting on the displaced blob. That force (buoyancy and gravitational) is $F = -(\rho^* - \rho')g$, since g acts downwards.

IN CLASS WORK

Use this force in Newton's second law and derive a simple equation of motion for the displacement δr. Show that a characterstic frequency N comes out

$$N^2 = \frac{g}{\rho *}\left(\frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = \frac{g}{\rho *}\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right], \tag{15.14}$$

called the Brunt-Väisälä frequency. Examine the solutions of the equation of motion based on the possible values of N in the stable or unstable condition.

Answer:
From Newton's second law

$$\rho^* \frac{\mathrm{d}^2 \delta r}{\mathrm{d}t^2} = -(\rho^* - \rho')g.$$

If we plug in Equation (15.10) we get the equation of motion

$$\frac{\mathrm{d}^2 \delta r}{\mathrm{d}t^2} + N^2 \delta r = 0,$$

where N is the Brunt-Väisälä frequency given above.
A general solution to this equation is $\delta r \propto \mathrm{e}^{\pm \mathrm{i} N t}$.
If the medium is stable to convection, we know that $N^2 > 0$. When this is the case, the solution is thus sinusoidal and the blob δr oscillates about a given point (gravity/buoyancy waves).
In the other case, $N^2 < 0$ and so N is imaginary: $N \to \mathrm{i}N$. The the solution goes as $\delta r \propto \mathrm{e}^{-Nt} + \mathrm{e}^{Nt}$. This solution describes an exponentially growing parcel, in other words, a convective instability.

Unit 16

Convection 2

16.1 Other formulations of the instability

- It is convenient to look at the stability criterion in terms of temperature gradients instead.
- Using Eq. (15.8) and the ideal gas law $P = \rho RT/\mu$ (ignore gradients in mean molecular weights FOR NOW) we can show that

$$\rho' = \rho + \frac{\rho}{p}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\delta r. \tag{16.1}$$

- For instability, again, we then require that $\delta\rho < 0$, or

$$\rho^* - \rho' = \left(\frac{1}{\gamma} - 1\right)\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\delta r < 0. \tag{16.2}$$

- Now the instability condition becomes

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} > \frac{\mathrm{d}T}{\mathrm{d}r}, \tag{16.3}$$

  where the adiabatic temperature gradient is given by

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{16.4}$$

- This says that if the temperature gradient decreases too steeply out through the star there will be convection.
- To simplify in analogy with what we did before (Eq. 15.13), it is convention to write the inequality as

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} > \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{16.5}$$

- We reintroduce

$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}, \qquad \nabla_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}, \tag{16.6}$$

  so that the instability condition is

$$\nabla > \nabla_{\mathrm{ad}}. \tag{16.7}$$

  Again, the RHS is 2/5 in the ionized ideal case.
- This is known as the **Schwarzschild criterion**. The inequality is also referred to as superadiabaticity.

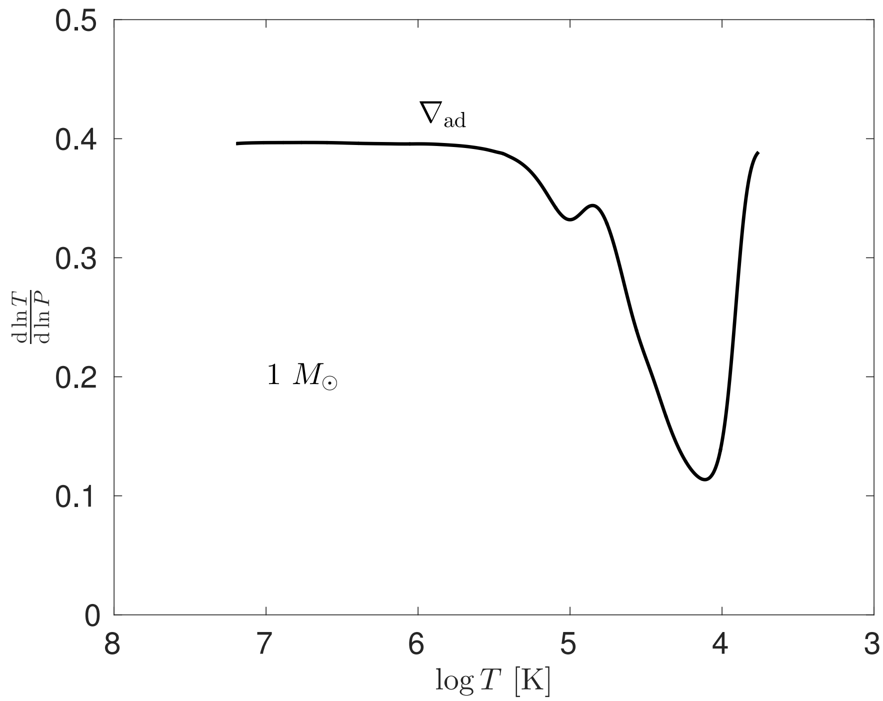

Figure 16.1: The adiabatic temperature gradient of a solar-like model. Note the decrease in the outer layers.

16.2 Physical conditions for convection onset

- Where in various types of stars does convection occur?
- Consider first energy transport by radiation. Recall Equation (15.3) which is reproduced here:

$$\nabla_{\rm rad} \equiv \left(\frac{d\ln T}{d\ln P}\right)_{\rm rad} = \frac{3}{16\pi acG}\frac{P\kappa}{T^4}\frac{L}{m},$$

- Another convenient form from substituting an equation of state is

$$\nabla_{\rm rad} \equiv \frac{3k_{\rm B}}{16\pi acGm_{\rm u}}\frac{\kappa}{\mu}\frac{L}{m}\frac{\rho}{T^3}. \tag{16.8}$$

- These are the gradients required to transport all the luminosity (L) by radiation. Keep in mind that $L = L(r)$ and $m = m(r)$.
- The radiative gradient $\nabla_{\rm rad}$ is a spatial derivative connecting P and T in consecutive mass shells.
- The adiabtic gradient $\nabla_{\rm ad}$ describes the thermal variation of one mass element upon an adiabatic compression.
- Anyway, for determining where an instability is, we must evaluate

$$\nabla = \nabla_{\rm rad} > \nabla_{\rm ad}, \tag{16.9}$$

  which, if satisfied, allows for convection to take place.
- So, a superadiabatic temperature gradient induces convective motions.
- The efficiency of convection (heat transport) increases when convection itself drives the temperature gradient very close to an adiabat.

- Let that sink in: Once convection is set up, it tends to drive ∇ very close to $\nabla_{\rm ad}$. So it goes from superadiabatic to nearly adiabatic.
- Examining the condition in Eq. (16.9), we see that convection may occur if
  - L/m is large. Think of this as $L/m \sim {\rm d}L/{\rm d}m = \varepsilon$ at small m (near the core). The energy generation rate is huge in massive stars in the cores, so many massive stars have convective cores.
  - κ is large. This is satisfied in the outer parts of less-massive stars, or with low surface temperatures and ionization zones of hydrogen (Sun).
  - ρ/T^3 is large. Happens typically in the outer parts of relatively cool stars. In fact, this ratio increases rapidly as the effective temperatures go down.
  - $\nabla_{\rm ad} = 1 - 1/\gamma$ is small. Satisfied in the ionization zone of hydrogen, in outer parts of cool stars where γ gets small (increased specific heat). See Figure 16.1.
- $n = 1.5$ polytropes are good models for convective regions, as this corresponds to $\gamma = 5/3$.

16.3 Depth of outer convection zones

- We've seen where convection tends to set in, but how large are these regions and what do they depend on?
- To first order the depth depends on $T_{\rm eff}$, since this determines where ionization layers are.
- Stars with lower surface temperatures achieve H ionzization at deeper depths with higher pressure.
- So for cool main-sequence stars, outer convection zones can extend deep, even to the center.
- For hotter stars, up to about 8000K, ionization occurs higher and higher in the atmosphere, and the convective shells of these stars can be very thin.
- The depth can also depend on **chemical abundances**.
- Consider a star with relatively high He abundance.
- The larger mean molecular weight corresponds to lower pressures at a given layer.
- He ionization, which can cause a convective instability, thus happens at a deeper layer, or higher temperature.
- Convection zones will be deeper for He-rich stars.
- For metal-poor stars, the opacity is reduced and therefore the radiative temperature gradient gets reduced, reducing the convective instability.
- Metal-poor stars will have shallower convection zones than metal-rich ones.

16.4 Semiconvection

- Let's return to the Brunt-Väisälä frequency again from Equation (15.14):

$$N^2 = g\left(\frac{1}{\gamma P}\frac{{\rm d}P}{{\rm d}r} - \frac{1}{\rho}\frac{{\rm d}\rho}{{\rm d}r}\right).$$

- We want to rewrite this in a very convenient form, and this time we **will** take into account composition gradients in the gas to be as general as possible. The form is

$$N^2 = \frac{g^2\rho}{P}(\nabla_{\rm ad} - \nabla + \nabla_\mu), \tag{16.10}$$

where (as some have been defined before)

$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}, \qquad \nabla_{\rm ad} = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\rm ad}, \qquad \nabla_\mu = \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P}. \tag{16.11}$$

EXAMPLE PROBLEM 16.1: Show how you can get from Equation (15.14) to Equation (16.10). Start with

$$\begin{aligned} N^2 &= g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right), \\ P &= \frac{\rho T}{\mu}, \\ \mathrm{d}\ln P &= \mathrm{d}\ln\rho + \mathrm{d}\ln T - \mathrm{d}\ln\mu. \end{aligned}$$

Answer:

$$\begin{aligned} N^2 &= g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right), \\ P &= \frac{\rho T}{\mu}, \\ \mathrm{d}\ln P &= \mathrm{d}\ln\rho + \mathrm{d}\ln T - \mathrm{d}\ln\mu, \\ N^2 &= g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}r} - \frac{\mathrm{d}\ln P}{\mathrm{d}r} + \frac{\mathrm{d}\ln T}{\mathrm{d}r} - \frac{\mathrm{d}\ln\mu}{\mathrm{d}r}\right), \\ &= g\left(\frac{\mathrm{d}\ln P}{\mathrm{d}r}\left(\frac{1-\gamma}{\gamma}\right) + \frac{\mathrm{d}\ln T}{\mathrm{d}r} - \frac{\mathrm{d}\ln\mu}{\mathrm{d}r}\right), \\ &= g\left(-\nabla_{\rm ad}\frac{\mathrm{d}\ln P}{\mathrm{d}r} + \frac{\mathrm{d}\ln T}{\mathrm{d}r} - \frac{\mathrm{d}\ln\mu}{\mathrm{d}r}\right), \\ &= g\frac{\mathrm{d}\ln P}{\mathrm{d}r}\left(-\nabla_{\rm ad} + \frac{\mathrm{d}\ln T}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\ln P} - \frac{\mathrm{d}\ln\mu}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\ln P}\right), \\ &= g\frac{\mathrm{d}\ln P}{\mathrm{d}r}\left(-\nabla_{\rm ad} + \nabla - \nabla_\mu\right), \\ &= \frac{g}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\left(-\nabla_{\rm ad} + \nabla - \nabla_\mu\right), \\ &= -\frac{g^2\rho}{P}\left(-\nabla_{\rm ad} + \nabla - \nabla_\mu\right), \\ N^2 &= \frac{g^2\rho}{P}\left(\nabla_{\rm ad} - \nabla + \nabla_\mu\right). \quad \square \end{aligned}$$

- Ignore the composition gradient for a second. We recover the "standard" stability relation: if the temperature gradient is larger than the adiabatic one (Schwarzchild), the BV frequency becomes complex.

- If it's the reverse, the BV is positive and the medium is stable to convection.

- Now however, we have the possibility that the Schwarzchild criterion is satisfied ($\nabla > \nabla_{\rm ad}$), yet the medium remains stable because the composition gradient makes it positive again.

- This is the *Ledoux criterion*, and when this is the case we have weak convection, or **semiconvection**.
- This typically would not occur in a convection zone, why? Because convection mixes material and composition gradients are removed.
- But in areas of nuclear burning where gradients do exist, and at the "edges" of convection zones, this situation can arise. Large peaks in the μ-gradient (also caused by g increases) can cause large jumps in N.
- Can be thought of as difficulty in moving "heavier" material (high μ) up - it doesn't want to do that.
- Becomes important for red-giant stars and their gravity modes mixing (boosting frequency) with acoustic modes.

16.5 Mixing length theory

- How does the convection actually transfer the energy?
- We have previously shown how convection can take place by considering a blob displaced from its equilibrium position.
- The blob is hotter than its surroundings.
- This blob rises with a velocity up to a point where a new hydrodynamic instability sets in, whereby the motion becomes turbulent and the blob dissolves, depositing its heat in the surroundings.
- The exact details of this process are still unknown and are a subject of much research. For solar convection, the most sophisticated numerical simulations carried out on the fastest computers take an order of magnitude more computation time than the solar "time" they are trying to simulate, and that is just for a small section of the Sun.
- In comparison, computing the solar model over the Sun's lifetime, with a simple model of convection, takes a couple minutes.
- Remember that the radiative flux $L/4\pi r^2$ is

$$F_{\rm rad} = \frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{{\rm d}T}{{\rm d}r} = \frac{4acG}{3}\frac{mT^4}{\kappa P r^2}\nabla,$$

  using hydrostatic equilibrium. ∇ is the actual stratification.
- Neither $F_{\rm rad}$ or ∇ are really known, since radiation only carries some of the star's flux.
- But we did note that the temperature gradient required to carry **all** of the stellar luminosity is $\nabla_{\rm rad}$. If this includes convection, the total flux then is

$$F_{\rm tot} = F_{\rm rad} + F_{\rm conv} = \frac{4acG}{3}\frac{mT^4}{\kappa P r^2}\nabla_{\rm rad}. \tag{16.12}$$

- A very reasonable convective flux for a parcel in pressure equilibrium is its heat content multiplied by the mass flux:

$$F_{\rm conv} = \rho v c_P \Delta T. \tag{16.13}$$

- All that's left to do is find ρ, v and ΔT of the blob.
- $\Delta T = T_i - T$ is the excess heat of a rising parcel of gas with respect to its surroundings.

- Assume a parcel moves a distance ℓ before dissolving into the background material, so that at any given time, a typical parcel will have moved $\ell/2$. The temperature difference between the parcel and the gas is
$$\Delta T = T_i - T = \left(\frac{\mathrm{d}T_i}{\mathrm{d}r} - \frac{\mathrm{d}T}{\mathrm{d}r}\right)\frac{\ell}{2}. \quad (16.14)$$

- Multiply by the pressure scale height $H_P = -\left(\mathrm{d}\ln P/\mathrm{d}r\right)^{-1}$ and divide by T
$$\frac{\Delta T}{T} = (\nabla - \nabla_i)\frac{\ell}{2H_P}. \quad (16.15)$$

- If the parcel remains in pressure equilibrium, assuming a general equation of state, then changes in its density are
$$\frac{\mathrm{d}\rho}{\rho} = -\delta\frac{\mathrm{d}T}{T} = -\delta(\nabla - \nabla_i)\frac{\ell}{2H_p}, \quad (16.16)$$
where
$$\delta = -\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln T}. \quad (16.17)$$

- Since the density difference implies a buoyancy force $-g(\rho_i - \rho)$, there is work that goes into moving the parcel
$$-g(\rho_i - \rho)\frac{\ell}{2} = g\rho\delta(\nabla - \nabla_i)\frac{\ell^2}{4H_p}. \quad (16.18)$$

- Suppose half of the work goes into the kinetic energy of the parcel, and the other half goes into the surroundings, moving it aside. The velocity is then
$$v^2 = \frac{\ell^2}{4}\frac{g\delta(\nabla - \nabla_i)}{H_p}. \quad (16.19)$$

- We get for the convective flux then that
$$F_{\mathrm{conv}} = \rho c_p T (g\delta)^{1/2}\frac{\ell^2}{4}H_p^{-3/2}(\nabla - \nabla_i)^{3/2}. \quad (16.20)$$

- It is possible to compute ∇_i, because as the parcel moves its internal energy will change because of radiative losses and adiabatic expansion/contraction. One finds
$$\frac{\nabla_i - \nabla_{\mathrm{ad}}}{\nabla - \nabla_i} = \frac{6acT^3}{\kappa\rho^2 c_p v\ell}. \quad (16.21)$$

- The overal problem can now be solved. Five new equations for the 5 unknowns F_{rad}, F_{conv}, v, ∇, and ∇_i. The *mixing-length parameter* $\alpha = \ell/H_P$ is used as a free parameter, chosen to match observations.

- Solutions can be found in the literature. Limiting cases are interesting. In dense regions of a star, $\nabla \to \nabla_{\mathrm{ad}}$. A gradient just over the adiabatic limit is all that is necessary to transport all of the luminosity.

- Near stellar photospheres, $\nabla \to \nabla_{\mathrm{rad}}$, and convection is ineffective so $F \to F_{\mathrm{rad}}$.

- In between these limits, the mixing-length equations need to be solved.

- Near the surface, convection does not transport energy efficiently, since convective flux is proportional to the the density which is quite low.

- It is also proportional to the mixing length which has to be small. So the only way for convection to dominate is if δ can get large. This means it has to be very far from adiabatic.

- We know this because we see *granulation*, which typifies the size of convective elements, about $10^{-3} R_\odot$.
- The convective time scale

$$t_{\rm conv} \simeq \left[\left(\frac{{\rm d} \ln T}{{\rm d} \ln r} \right)_{\rm ad} - \frac{{\rm d} \ln T}{{\rm d} \ln r} \right]^{-1/2} t_{\rm dyn}. \tag{16.22}$$

- This timescale is a dynamical time scale, but since the gravitational acceleration is reduced because of the difference in density that comes into all of this, the timescale is increased by that pre-factor.
- That prefactor in the Sun is of order 10^{-6}, and we find convective time scales $t_{\rm conv} \simeq 0.2$ yr.
- This is still small compared to evolution. Matter mixes over such a time scale in the convection zone.

16.6 Convective overshoot

- We briefy mention this here because it has very important implications for stellar modeling.
- The parcels we consider unstable to convection will eventually reach a stable layer.
- However the momentum causes them to overshoot the boundary into this stable layer.
- For example, this is why we see granulation in the solar photosphere (which is stable).
- One important consequence of this overshoot is the deposition of convectively mixed material into stable regions, which can affect the thermal structure as well as evolutionary properties.

16.7 An entropy formulation

- Convection can also be understood in terms of entropy.
- For reversible processes, $\mathrm{d}Q = T\mathrm{d}S$ and so

$$T\,\mathrm{d}S = \mathrm{d}U + P\mathrm{d}V. \tag{16.23}$$

- Using standard thermodynamic relations, it can be shown from here that

$$\frac{\mathrm{d}S}{\mathrm{d}r} = c_P(\nabla - \nabla_{\rm ad}) \frac{\mathrm{d}\ln P}{\mathrm{d}r}. \tag{16.24}$$

- So if the star is radiative, $\mathrm{d}S/\mathrm{d}r > 0$ and the entropy increases outward.
- If the star is convective, $\mathrm{d}S/\mathrm{d}r < 0$. If the convection is efficient the gradient is very close to being adiabatic, meaning the entropy is very nearly constant throughout convection zones.

16.8 An energy formulation

- Start with Equation (16.4) and rewrite using ideal gas law and hydrostatic equilibrium:

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\rm ad} = \left(1 - \frac{1}{\gamma}\right) \frac{\mu}{\rho R_g} \frac{\mathrm{d}P}{\mathrm{d}r} = -\left(1 - \frac{1}{\gamma}\right) \frac{g\mu}{R_g}. \tag{16.25}$$

- Remembering that $\gamma = c_P/c_V$ and $c_P - c_V = R_g/\mu$,

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = -\left(\frac{c_P/c_V - 1}{c_P/c_V}\right)\frac{g\mu}{R_g}, \tag{16.26}$$

$$= -\left(\frac{\frac{c_P - c_V}{c_V}}{c_P/c_V}\right)\frac{g\mu}{R_g}, \tag{16.27}$$

$$= -\left(\frac{c_P - c_V}{c_P}\right)\frac{g\mu}{R_g}, \tag{16.28}$$

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = -\frac{g}{c_P}. \tag{16.29}$$

- This form shows how the parcel of gas is changing as it rises adiabatically and expands

- This can also be derived using energy by considering energy release and work against gravity

- To see this, consider again a parcel of gas at position r that is displaced adiabatically to $r + \mathrm{d}r$. The temperature at the new depth is $T + \mathrm{d}T$.

- The temperature gradient $\mathrm{d}T/\mathrm{d}r$ will be negative in our case since T decreases as we move further from the center. So when the parcel arrives at the new location, it is in cooler surroundings, so the parcel has more internal energy than the local gas.

- If we let time elapse a bit and the parcel releases energy to the surroundings, the amount of thermal energy released at fixed pressure is $C_p\,\mathrm{d}T$ erg g^{-1}. Is this energy release important?

- The other factor at work here is gravity, and we have to ask if the work done against gravity to move the parcel upwards balances the release of thermal energy.

- The work done against gravity is $g\,\mathrm{d}r$ per gram of material. Therefore, the total amount of energy to displace 1 gram of material is

$$\Delta E = g\,\mathrm{d}r + C_p\,\mathrm{d}T. \tag{16.30}$$

  Note that the sign can be either positive or negative (since $\mathrm{d}T$ will usually be negative). Let's say that the gravitational energy exceeds the thermal energy released, so that the change in energy is positive. In this case, it has taken more work to displace the gas than what is released as thermal energy, and so the final energy state is higher. Therefore, the parcel will sink down to its initial position in a low energy state. There is no reason for it to move. Is is *convectively stable.*

  In the opposite case, the release of thermal energy is more than enough to compensate for the work against gravity. There will even be some excess, and the parcel will still be hotter than its surroundings. It may continue on rising. This is a *convective instability*, as we have already studied.

- So, the boundary between the two cases is when $g\,\mathrm{d}r = -C_p\,\mathrm{d}T$, which occurs when the temperature gradient takes on a particular value of the *adiabatic temperature gradient*

$$\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}} = -\frac{g}{C_p}. \tag{16.31}$$

  For convection to occur, $\mathrm{d}T/\mathrm{d}r < \mathrm{d}T/\mathrm{d}r|_{\mathrm{ad}}$. Since both gradients are negative, this can also be written for instability

$$\nabla > \nabla_{\mathrm{ad}}. \tag{16.32}$$

  This is the same conclusion we reached earlier.

COMPUTER PROBLEM 16.1: [80 pts]: This assignment deals with convection and radiation zones in main-sequence stars over a range of different masses. You are going to quantitatively illustrate where we expect such zones to exist in stars to provide an overall understanding of stellar structure during the period of core-hydrogen burning.

What to do

Your overall goal is to prepare a main plot (described in more detail below) showing where these zones exist. For this purpose, you need to run a bunch of `MESA` models for different mass stars. We will use these models for future studies so make sure to keep the directory(ies) neat and handy.

1. The models should be within the stellar mass range of $M_* = 0.3\,M_\odot - 20\,M_\odot$. The number of models you compute is up to you, but the "mass grid" should be sufficiently populated so you don't miss any important details, and so that you can make the required plots with enough number of points and precision. Somewhere between 10 and 20 models seems reasonable. One of the models should be $M_* = 1\,M_\odot$, of course!
2. The models are to be of solar hydrogen mass fraction and metallicity ($Z = 0.02$). For each mass, you will run the code until the core looks like the core of today's Sun, $X \approx 0.35$, which is the model you will want to use for what follows. You will also want the history data (really just $T_{\rm eff}$ and L) for the ZAMS (zero-age main sequence) point, where $X \approx 0.7$. You shouldn't need to run pre-main sequence models.
3. That's all you need to do with `MESA`. Now you need to carefully explore the resulting data and determine the locations of all the convection and radiation zones for each star mass. Use the criteria discussed in class.

What to hand in

After reading the below information you will find you need to hand in at least 7 plots, as well as text pertaining to items 4 and 6.

1. A nice H-R diagram of your stars. The symbol size should represent the star's radius, and the color the star's mass at the time you stopped your models. Also give an indication of age somehow discretely so we have an idea of the range there. Finally, plot a small symbol for where each star began on the ZAMS, so we see how much "evolution" occured. These symbols do NOT need to show the mass or radius information as in the more evolved points. [15 pts]
2. A plot that shows where the internal convection and radiation zones are for each star. [25 pts]
   - This should be a publication-quality plot (clearly labeled, easy to read, useful for future reference, etc.).
   - The plot is to be as follows: the x axis is $\log M_*/M_\odot$ and the y axis is $m(r)/M_*$.
   - The interior radiation and convection zones should be clearly demarcated by however you wish to do it.
   - Finally, somehow denote the dominant H-burning process for everywhere that nuclear burning is taking place. In many cases, you will have some regions where PP is dominant and some where CNO is (in the same star).
3. A second plot of exactly the same information but instead where r/R_* is the ordinate. These should be on separate pieces of paper. [5pts]
4. A summary paragraph or two of how you obtained the parameters in your plot, i.e., how you determined where the zones are from the stellar profiles. Mention here any issues you ran into or anomalous cases (if any). You don't need to do much interpretation here, just write down what you did to gather the information that went into these plots. [10pts]

   Answer:
   To determine where the zones are, one simply needs to compare where $\nabla_{\rm R} > \nabla_{\rm ad}$. This defines where a convective instability will occur. Figure 16.2 shows this plot.

5. Pick two of your models. One should be $M_* \leq 1.1 M_\odot$ and one $M_* \geq 3 M_\odot$. For each of these 2 models you will make 2 more plots, so 4 in total here. One of the plots for each mass should have the three quantities ∇, $\nabla_{\rm ad}$, and $\nabla_{\rm rad}$ plotted as a function of interior $\log T$, but reversed, so $\log T$ decreases to the right. The other plot for each mass will be the Brunt-Väisälä frequency (squared), or N^2 in appropriate units as a function too of reversed $\log T$. Make sure it's scaled properly as to be visible. [10pts]

   Answer:

   Figures 16.3 and 16.4 show the profiles for a $1 M_\odot$ and a $6 M_\odot$ model, respctively.

6. Write another page or two on the above 4 plots. Describe what is happening and connect what you observe in these figures to the findings shown in your **main** plot. Compare and contrast the behavior for the 2 different masses. Try to describe and decipher any unusual "features" you see, such as bumps or spikes, in each case. Connect them to things we may have discussed in class, not forgetting, for example, our study of the opacities. Indeed, you may find it necessary to generate other supplementary plots of other quantities in the models to get your point across here. Feel free to draw on and label these plots by hand to point things out. Be thorough. Also in this part, be sure to answer the following question for each of these 2 stellar mass models:

   - In the interior regions where $\nabla_{\rm rad}$ is large, what is your idea as to the physical reasons causing this?

   Use Equation (16.8) from the notes as a guide if needed. [15 pts]

   Answer:

   - In the outermost part of the $1 M_\odot$ star, convection is very inefficient and $\nabla \approx \nabla_{\rm rad}$. But $\nabla_{\rm ad}$ is not constant because of partial ionization zones. The dips in the gradient are labeled.
   - The large values of $\nabla_{\rm rad}$ are due to large opacities in the convection zone. These opacities are because of appreciable H ionization as one goes deeper. Remember, in terms of mass, this zone is very small, but it does span 2 orders of magnitude in temperature.
   - In the $6 M_\odot$, the reduction in the gradient because of He ionization is so small that it is doubtful that convection can really set in here. Hydrogen is basically already ionized. The core has extreme radiation.
   - The BV frequencies are only finite where radiation is the energy transport.
   - To stabilize convection outside of the core of massive stars, because of overshoot, it is proper to use the Ledoux criterion instead of the Schwarzschild one. Figure 16.5 shows two cases, which indeed illustrates the stabilization.

Things to keep in mind

1. Your main plot is going to be judged by the faculty to determine who made the best publication-quality image (oh, and it should be correct too). Therefore the plot should be full page and nice, and on a separate page from your other stuff! You also may want to keep it secret from your classmates so they don't steal your ideas. An emailed .pdf or .eps of your plot would also be good and I can collect them all into one document to show in high quality on a computer screen. The one judged best will receive a special prize extra credit on the assignment.
2. You may want to consider your initial stellar mass grid spaced out regularly in terms of $\log M_*/M_\odot$, rather than in linear $M_*/M_\odot$. But do what you want. Much of the changes to convection properties occur in the $1-2\ M_\odot$ range.
3. You may need to consider adding quantities to what `MESA` saves in the profiles. This is easily done using the .list files. Your best bet might be copy the default profile_columns.list file into your working directory and uncomment the things you'll need. If you do that, comment out the line in your inlist_project that just uses the custom_profiles.list. Let me know if you need help.
4. You likely don't need to create any pre-main-sequence models because the metallicity we are using is standard.

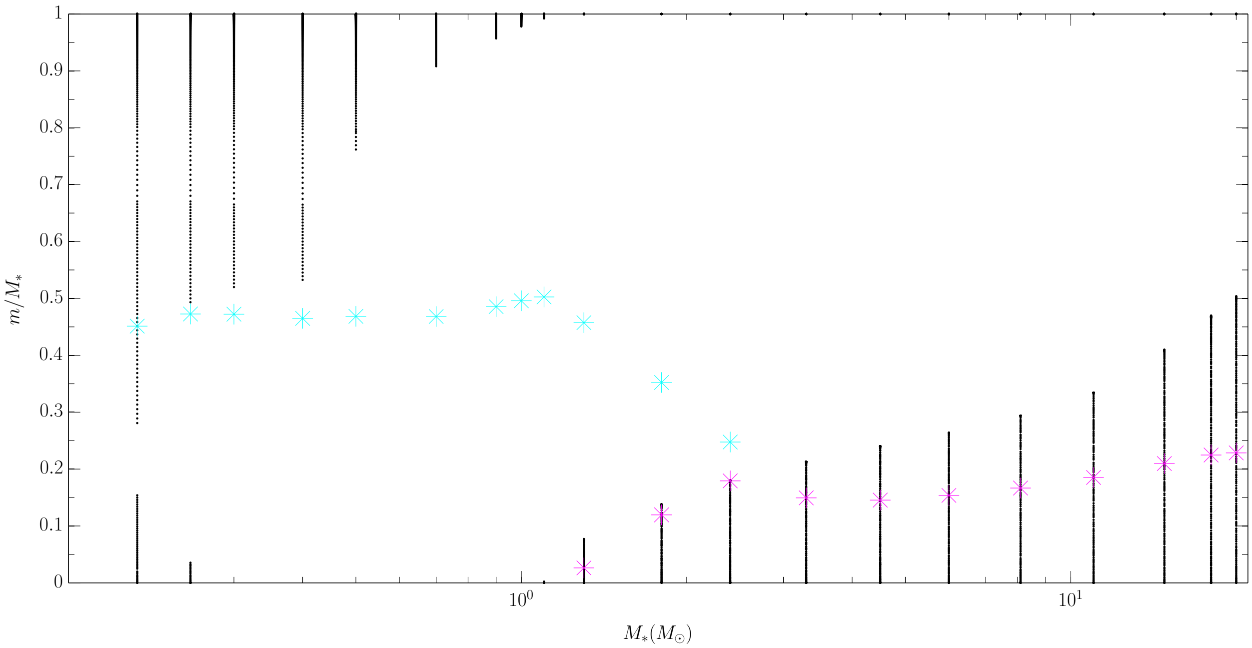

Figure 16.2: Convective and radiative regions. The black dots represent convection zones. A core in each star is defined as where 99% of the luminosity is generated. The cyan stars represent the PP-dominated burning regions within the core down to the center or down to the magenta stars if the CNO cycle dominates at a given mass.

5. It may or may not be reasonable to interpolate certain quantities to make your mass grid denser for your plots. It's up to you.
6. There are very fast and easy, or long and complicated ways of obtaining the profile quantities you need for the plots. Hopefully you find the fast/easy ways.
7. Try to understand what's going on in each model. This will be helpful for grasping the evolution of stars of different masses.

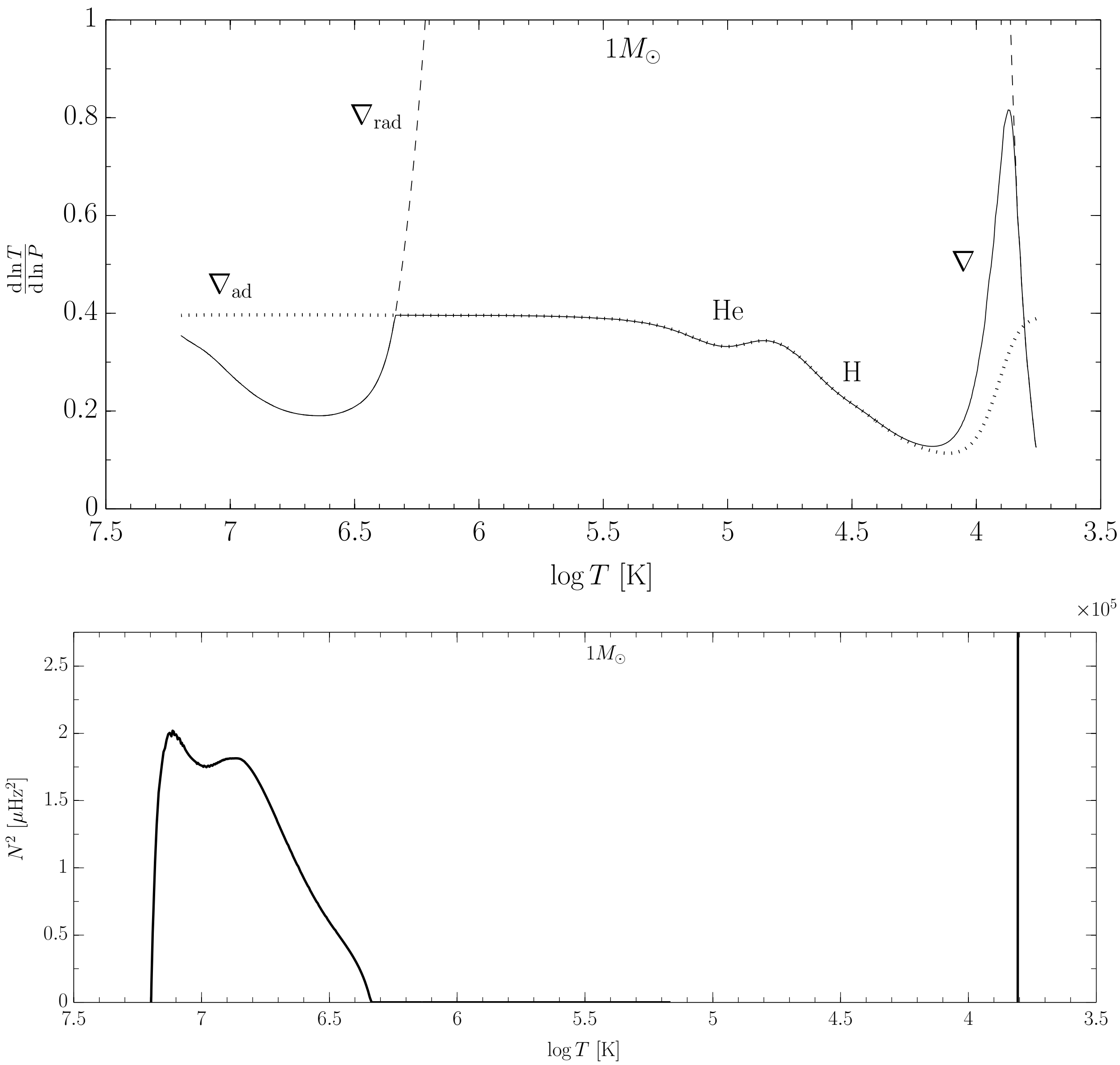

Figure 16.3: The lower mass model.

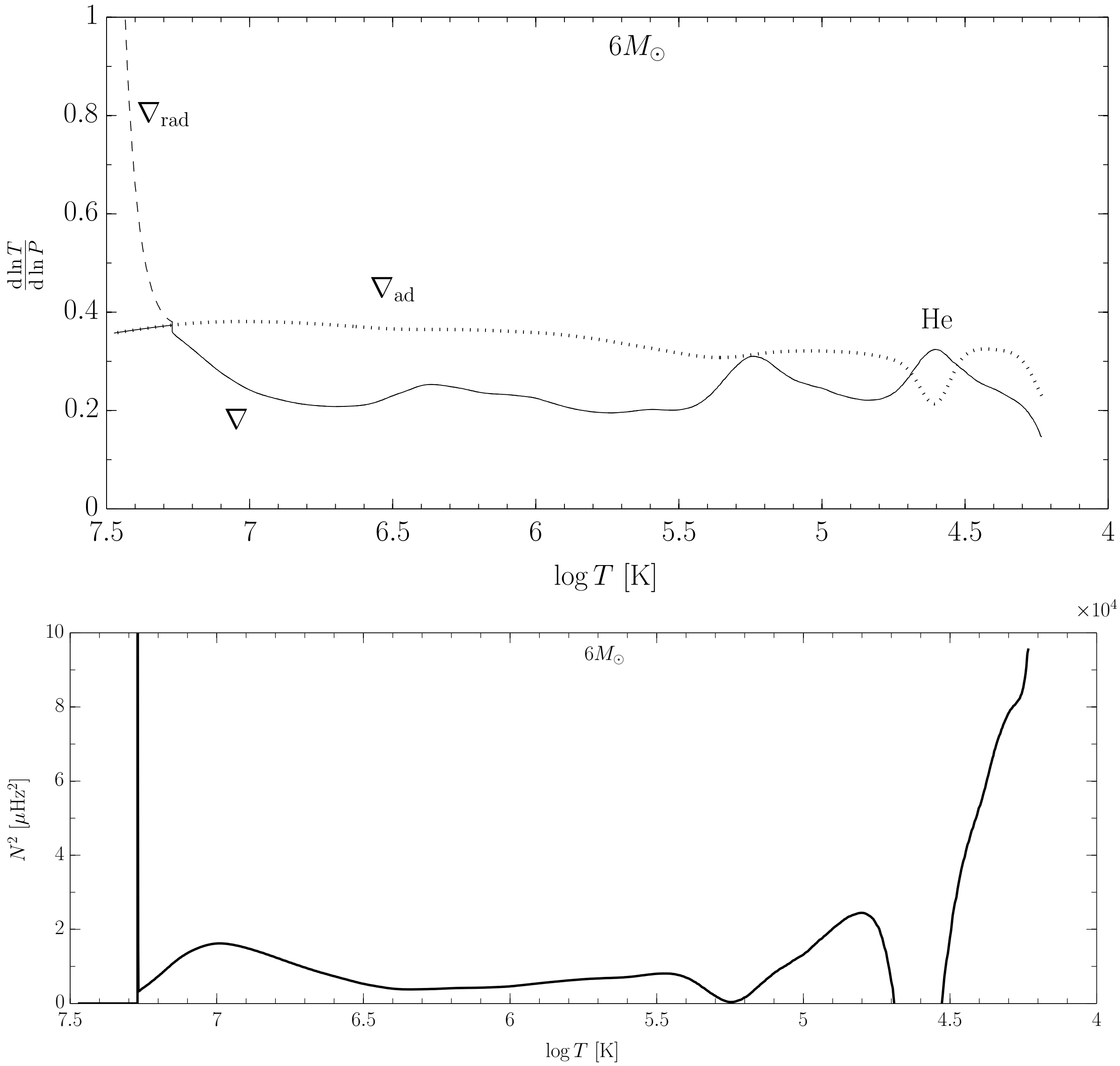

Figure 16.4: The higher mass model.

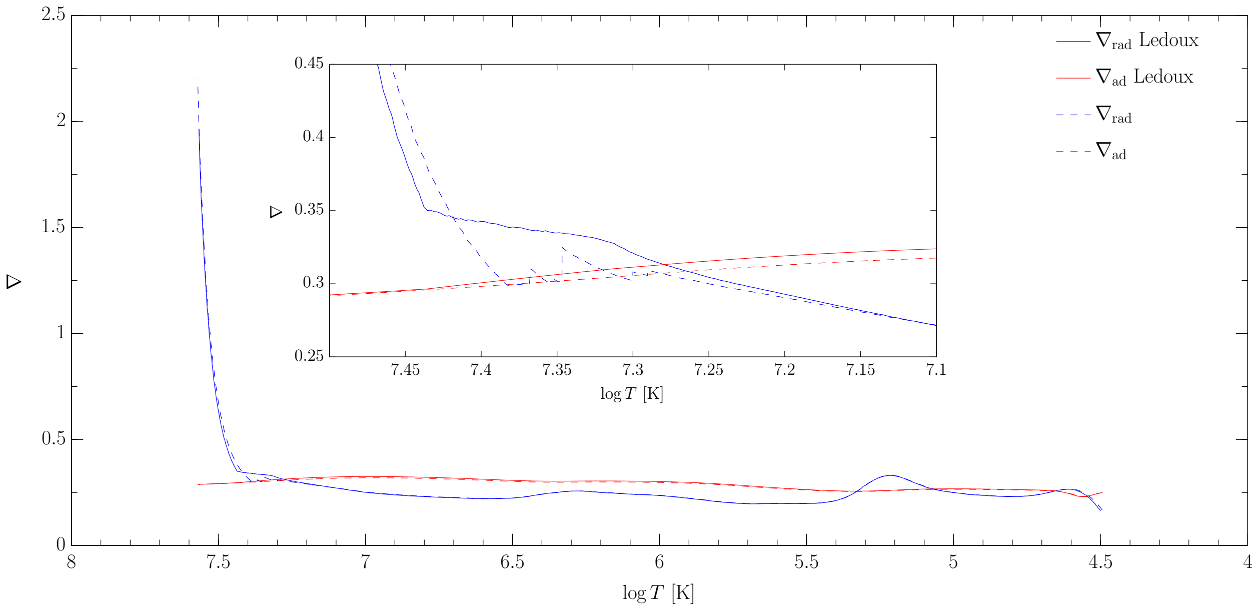

Figure 16.5: Comparison of using the Ledoux criterior for convection and not. The solid curves are the radiative and adiabatic gradients when compositional changes are taken into account. The dashed lines use standard Schwarschild criteria. In the zoomed-in region, one sees that the convective instability is surpressed, as one might expect.

Part II

Stellar Evolution

Unit 17

Overview of Asteroseismology

- The surface of a star oscillates with displacements in the radial and horizontal directions as:

$$\boldsymbol{\delta r} = \xi_r \hat{\boldsymbol{r}} + \boldsymbol{\xi}_\mathrm{h},$$

- The diplacements are solutions to the equations of motion expressed as

$$\xi_r(r, \theta, \phi, t) = \sqrt{4\pi}\, \tilde{\xi}_r(r) Y_\ell^m(\theta, \phi) \exp(-i\omega t) \tag{17.1}$$

  and

$$\boldsymbol{\xi}_\mathrm{h} = \sqrt{4\pi}\tilde{\xi}_\mathrm{h}(r) \left(\frac{\partial Y_\ell^m}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta} \frac{\partial Y_\ell^m}{\partial \phi} \hat{\boldsymbol{\phi}} \right) \exp(-i\omega t). \tag{17.2}$$

- The r dependence terms are the (as of yet undetermined) eignenfunctions of the star, while ω is the angular eigenfrequency of a mode
- The Y_ℓ^m are spherical harmonics
- Three "quantum" numbers describe each mode: n gives the overtone, or number of nodes in the radial direction
- ℓ is the degree and specifies the number of surface nodes
- m is the azimuthal order and $|m|$ gives the number of nodal lines of longitude
- The equations of motion are derived from hydrodynamics (continuity, momentum, energy, Poisson, etc.) and approximations to them
- Namely 2: adiabatic small-amplitude (linear) changes
- After the approximations and plugging the spherically symmetric expansions into them, a set of coupled differential equations are derived
- These are then solved for the eigenfrequencies and eigenfunctions
- The pulsations can be classified in various ways
- **Radial modes** are the simplest with $\ell = 0$, these are the breathing modes.
- The star expands and contracts, heats and cools. The fundamental mode is what is foundin Cepheid and RR Lyrae stars.
- The center is a node and the surface is an antinode

- The first overtone $n = 2$ has 1 node away from the center.
- The motions above and below that region move in antiphase
- When stars exhibit both fundamental and first overtone pulsations, the ratio of their frequencies can be used to determine interior properties
- Unlike musical instruments, their ratios vary, which at least tells us that stars do not have constant temperatures or uniform sound speed
- **Nonradial modes** have $\ell > 0$ and $n \geq 1$, thus, a surface node
- These modes are degenerate in m as there are $2\ell + 1$ modes with the same frequency
- Rotation and possibly other effects can break this degeneracy
- The simplest is the dipole $\ell = 1$ mode
- The equator is a node, so the northern hemisphere will expand while the southern contracts, and vice versa
- Depending on n there are internal radial nodes as well
- $\ell = 2$ are quadrupolar, while $\ell = 3$ are octupolar modes
- Beyond this, except for the Sun, we cannot detect higher-degree modes, as the surface averaging partially cancels neighboring regions of hot/cold (intensity) or up/down (Doppler velocity)
- Nodal regions beyond about $\ell = 3$ are practically unattainable in disc-integrated observations (including for the Sun)

Unit 18

Theory of the Main Sequence

18.1 Summary of stellar structure

- Mass and radius relationship
$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho$$
- Hydrostatic equilibrium
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$$
- Energy
$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho\varepsilon$$
- Energy transport
$$L = 4\pi r^2 (F_{\mathrm{rad}} + F_{\mathrm{conv}})$$
  - Radiation
$$F_{\mathrm{rad}} = -\frac{4acT^3}{3\kappa_{\mathrm{R}}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$
  - Convection
$$F_{\mathrm{conv}} = \rho C_p T (g\delta)^{1/2} \frac{\ell^2}{4} H_p^{-3/2} (\nabla - \nabla_{\mathrm{ad}})^{3/2}.$$
- Equation of state
$$P = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}} + \frac{a}{3} T^4$$
- Rosseland opacity
$$\kappa_{\mathrm{R}} = \kappa_{\mathrm{R}}(\rho, T, \mu)$$
- Energy generation
$$\varepsilon = \varepsilon(\rho, T, \mu) = \varepsilon_{\mathrm{nuc}} + \varepsilon_{\mathrm{grav}}$$
  where
$$\varepsilon_{\mathrm{nuc}} = \varepsilon_0 \rho T^n, \tag{18.1}$$
  and
$$\varepsilon_{\mathrm{grav}} = -T\frac{\mathrm{d}S}{\mathrm{d}t}$$
  is related to Kelvin-Helmholtz contraction (and note the time dependence).

- Abundance changes

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= -\frac{\varepsilon_{\mathrm{nuc}}}{26.7\,\mathrm{MeV}} \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= -\frac{\mathrm{d}X}{\mathrm{d}t}\end{aligned}$$

- Boundary conditions

$$\begin{aligned} r &\longrightarrow 0; \quad m(r) \longrightarrow 0 \\ r &\longrightarrow 0; \quad L(r) \longrightarrow 0 \\ r &\longrightarrow R; \quad m(r) \longrightarrow M \\ r &\longrightarrow R; \quad \rho(r) \longrightarrow 0 \\ r &\longrightarrow R; \quad T(r) \longrightarrow T_{\mathrm{eff}} \end{aligned}$$

- Vogt-Russell Theorem
  - There are only 2 free parameters in the equations needing to be solved: the total stellar mass M and the chemical composition.
  - The theorem states that just these 2 parameters uniquely determine the structure of any star. Then, evolution is only based on the changing of the composition due to nuclear burning.

18.2 Homology relations for stars in radiative equilibrium

- Solving the equations of stellar structure is possible, yet difficult.
- There are ways of obtaining useful insights without doing so (like using the first 2 equations to study polytropes).
- This involves basic unit analysis.
- Let's take a simplified subset of the equations written above and express them in terms of the mass coordinate, rather than position:

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi\rho r^2}, \tag{18.2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}, \tag{18.3}$$

$$\frac{\mathrm{d}F}{\mathrm{d}m} = \varepsilon_0 \rho T^n, \tag{18.4}$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = -3\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}, \tag{18.5}$$

$$P = \frac{R_g \rho T}{\mu}. \tag{18.6}$$

- We've neglected things like radiation pressure and taken simple opacities.
- Now introduce the dimensionless mass fraction parameter

$$x = \frac{m}{M_*}, \tag{18.7}$$

so that our functions may be expressed now in terms of dimensionless functions of x:

$$
\begin{aligned}
r &= f_1(x)R_*, & (18.8)\\
P &= f_2(x)P_*, & (18.9)\\
\rho &= f_3(x)\rho_*, & (18.10)\\
T &= f_4(x)T_*, & (18.11)\\
F &= f_5(x)F_*. & (18.12)
\end{aligned}
$$

- Now, plugging in these scaling equations Eqs. (18.8)-(18.12) into Eqs. (18.2)-(18.6) gives

$$
\begin{aligned}
\frac{\mathrm{d}f_2}{\mathrm{d}x} &= -\frac{x}{4\pi f_1^4}; \qquad P_* = \frac{GM_*^2}{R_*^4}, & (18.13)\\
\frac{\mathrm{d}f_1}{\mathrm{d}x} &= \frac{1}{4\pi f_1^2 f_3}; \qquad \rho_* = \frac{M_*}{R_*^3}, & (18.14)\\
f_2 &= f_3 f_4; \qquad T_* = \frac{\mu P_*}{R_g \rho_*}, & (18.15)\\
\frac{\mathrm{d}f_4}{\mathrm{d}x} &= -\frac{3f_5}{4f_4^3(4\pi f_1^2)^2}; \qquad F_* = \frac{ac}{\kappa}\frac{T_*^4 R_*^4}{M_*}, & (18.16)\\
\frac{\mathrm{d}f_5}{\mathrm{d}x} &= f_3 f_4^n; \qquad F_* = \varepsilon_0 \rho_* T_*^n M_*. & (18.17)
\end{aligned}
$$

- The equations above on the left are a set of nonlinear differential equations of dimensionless quantities that are **independent of the total mass**.
- The *dimensional* coefficients of the $f_i(x)$ functions, i.e., T_*, P_*, etc., are obtained as functions of mass by solving the algebraic equations on the right hand side of the above equations.
- So combining the solutions for each of these 2 procedures gives us physical profiles of quantities for any M_*.
- The key point is that the shapes of the profiles as a function of x is the same for all stars. They only differ by a constant factor determined by the mass.
- This is called *homology* relations.

18.3 Dependence on mass

- The main sequence gives a tight relationship between luminosity and effective temperature, and one would like to understand this physically.
- The slope can be expressed as

$$\log L = \alpha \log T_{\rm eff} + \text{const.} \tag{18.18}$$

- Another important property is the correlation of luminosity on mass:

$$L \propto M^{\nu}. \tag{18.19}$$

- By solving the algebraic equations given in Eqs. (18.13)-(18.17), we can ascertain the dependence of physical properties on the stellar mass.
- By plugging the density and pressure into the temperature expression, one obtains

$$T_* = \frac{\mu G}{R_g}\frac{M_*}{R_*}, \tag{18.20}$$

and using this in the first flux equation, we find

$$F_* = \frac{ac}{\kappa}\left(\frac{\mu G}{R_g}\right)^4 M_*^3. \tag{18.21}$$

- This is an important result. At any interior point a star 10 times more massive than another will have 1000 times the flux.
- At the surface, therefore, the luminosity is

$$L \propto M_*^3, \tag{18.22}$$

which is what we set out to find.

- Using some of these results and substituting into the previous ones, we also find

$$R_* \propto M^{\frac{n-1}{n+3}}. \tag{18.23}$$

- This relates a star's radius to a star's mass depending on details of the energy generation.
- Let's consider $n \approx 4$ for proton chain stars (lower mass), and $n \approx 16$ for CNO cycle stars (higher mass).

$n = 4$	$n = 16$
$R \propto M^{3/7}$	$R \propto M^{0.8}$

- So in massive stars, the radius is proportional to the mass, whereas the mass dependence is weaker in less massive stars. But the relationship is not inverse like we saw for compact objects.
- Using this result and considering the density gives

$$\rho_* \propto M^{2\frac{3-n}{3+n}}. \tag{18.24}$$

- Since n is always larger than 3, this results shows that lower-mass stars are denser than higher-mass stars. We've determined this already from other means.
- Now for the main sequence on the H-R diagram, the radius may be eliminated from $L = 4\pi\sigma R^2 T_{\text{eff}}^4$ using what we just found to give

$$L^{1-\frac{2(n-1)}{3(n+3)}} \propto T_{\text{eff}}^4. \tag{18.25}$$

- This gives

$n = 4$	$n = 16$
$\log L = 5.6 \log T_{\text{eff}} + c$	$\log L = 8.4 \log T_{\text{eff}} + c$

- This is in rough agreement with observations, except for the very low main sequence (where stars are fully convective).
- Many other interesting quantities can be computed in this way. An important one is the main sequence lifetime dependence on mass. A reasonable assumption is that the nuclear resevoir depends on mass, and the rate of consumption is related to the luminosity. So

$$\tau_{\text{MS}} \propto \frac{M}{L} \propto M^{-2}. \tag{18.26}$$

- We can estimate a minimum mass on the main sequence too.

- According to what we found already, the temperature

$$T_* \propto \frac{M_*}{R_*} \propto M^{\frac{4}{n+3}}. \tag{18.27}$$

- This holds at the center as well. For a low-mass star

$$T_c \propto M^{4/7}. \tag{18.28}$$

- The minimum temperature for hydrogen burning is about 4 million K. We know the Sun is doing this, so let's scale this equation to the Sun:

$$\frac{T_c}{T_{c,\odot}} = \left(\frac{M}{M_\odot}\right)^{4/7}. \tag{18.29}$$

- Since $T_c \geq T_{\rm min}$, we can set the condition that

$$\frac{M}{M_\odot} \geq \left(\frac{T_{\rm min}}{T_{c,\odot}}\right)^{7/4}. \tag{18.30}$$

- This gives about $M_{\rm min} \approx 0.1 M_\odot$ using the Sun's core temperature.
- The luminosity at this minimum mass, estimated similarly, is

$$\frac{L_{\rm min}}{L_\odot} = \left(\frac{M_{\rm min}}{M_\odot}\right)^3 \approx 10^{-3}. \tag{18.31}$$

Unit 19

Evolution on the main sequence

19.1 Low-mass stars

- The time of arrival on the main sequence is known as the ZAMS - zero-age main sequence.
- "Where" it ends up depends only on mass and chemical mixture.
- The lower mass limit is roughly about $0.1M_\odot$.
- The upper mass limit is about $100M_\odot$.
- The mean molecular weight changes a lot. Consider fully ionized H in core at beginning at ZAMS (see Equation (5.10)):
$$\mu = \frac{4}{3 + 5X - Z} \simeq 0.61. \tag{19.1}$$
- As all of it gets converted to He, we then have
$$\mu = \frac{4}{3 + 5(0) - Z} \simeq 1.3. \tag{19.2}$$
It more than doubles!
- This change (increase) in mean molecular weight causes changes in other things. It can be shown from homology that the hydrogen-burning luminosity $L_\mathrm{H} \propto \mu^7$.
- Also note that the opacity is reduced, as He is less opaque than H under the same conditions.
- The number of free particles also decreases, as does the pressure.
- A low-mass star slightly contracts its core and heats up. Firstly, ρ increases by the core contracting. As this happens gravitational energy gets released according to the Virial theorem, which partly goes to increasing the thermal energy of the core - increased T.
- This must increase the pressure to account for the "heavier" material. Indeed, according to the equation of state, if P increases, and μ increases, then ρT must certainly increase.
- The nuclear energy generation increases and then so does the luminosity.
- This causes a slow increase of the star's luminosity over the whole MS phase.
- Also, as the core contracts, the surface radius increases, slightly for low-mass stars, more rapidly for high-mass stars. This can affect the effective temperature (see below).

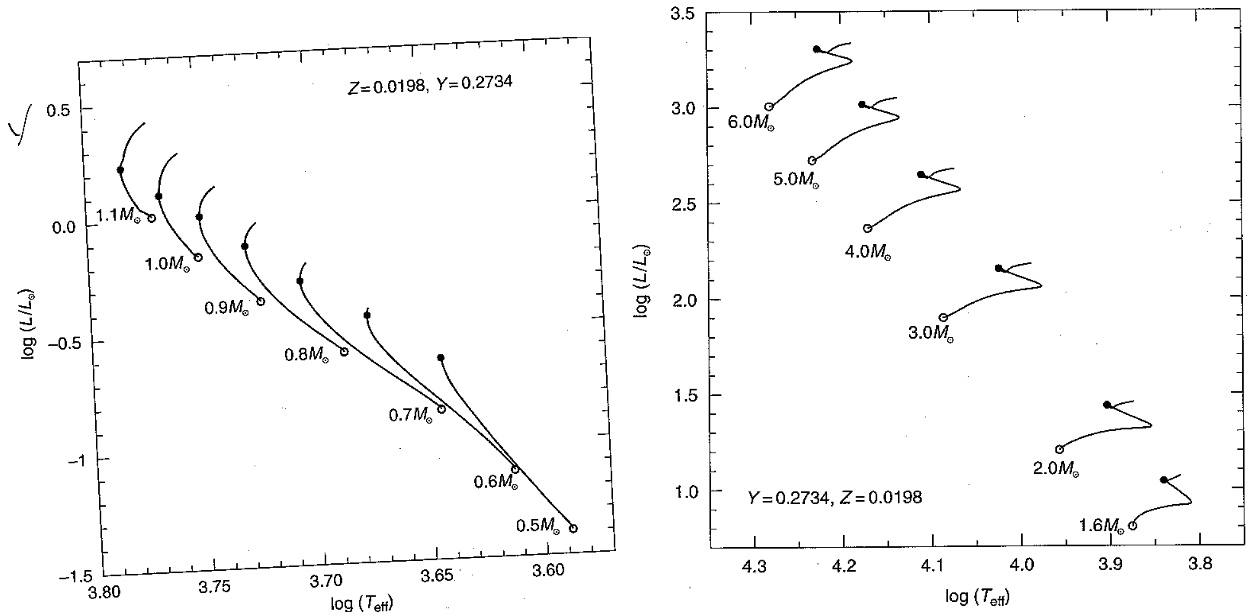

Figure 19.1: (Left): Low-mass star evolution on the main sequence. (Right): The same but higher-mass stars.From Salaris and Cassisi [2006].

- See Figure 19.1 for the MS evolution.
- Figure 19.2 (left) shows the core hydrogen mass fraction as a function of time. The burning region extends out to a significant radial distance.
- How much has the solar luminosity changed over time? Calculations (including homology ones) show that the Sun's luminosity compared to its ZAMS luminosity L_0 is $L \simeq 1.26L_0$. This means it was about 25% less luminous than it is today, which has/had implications for the Earth.
- All of the main properties of the Sun from its ZAMS point until today are shown in Figure 19.3.
- The important things to note are the increase in density and decrease in X. One would think the temperature increase too would really increase the nuclear energy generation rate ϵ_c.
- But as we saw (Equation (3.9)), it not only depends on T but also on X^2, so it is somewhat halted by the decreasing hydrogen amount over time.

19.2 High-mass stars

- The main difference in these stars is the increased temperature in the core, as in Figure 19.5.
- Thus, the CNO cycle is the dominant luminosity source.
- This has the effect of concentrating the the luminosity production in the inner 10% of the mass for a $10M_\odot$ star, compared to about 30% for a $1M_\odot$ star.
- The other effect is a steep temperature gradient in the inner regions due to the high flux. Thus, convection kicks in.
- This region becomes fully mixed chemically, as in Figure 19.2, right.
- The outer regions are radiative, since the ionization regions are very far out in the atmopshere compared to low-mass stars.

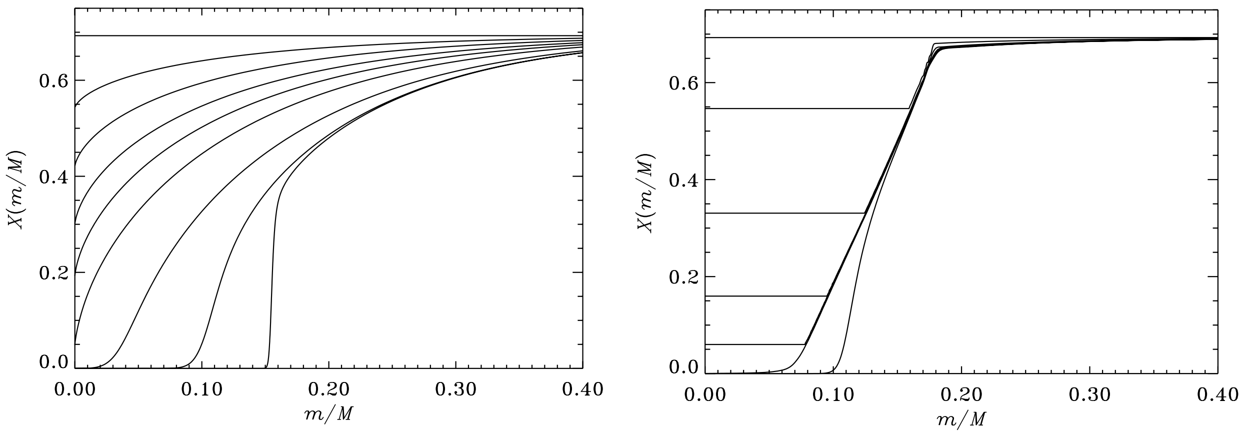

Figure 19.2: (Left): Hydrogen profiles showing the gradual exhaustion of hydrogen in a $1M_\odot$ star. The homogeneous initial model consists of a mixture with a hydrogen abundance by mass of 0.699. X as a function of the mass fraction $m/M_\odot$ is plotted for nine models which correspond to ages of 0, 2.0, 3.6, 5.0, 6.2, 7.5, 9.6, 11.0 and 11.6 times 10^9 years, after the onset of hydrogen burning. The model at 5.0×10^9 years corresponds roughly to the present Sun, whereas the last two models are in the shell hydrogen burning phase. (Right): The same but for a $2.5M_\odot$ star. The lines show the hydrogen profiles for models of age 0, 1.5, 3.1, 4.0, 4.4, 4.6, and 4.8 times 10^8 years. From Christensen-Dalsgaard [2008].

- As evolution occurs, the star gets brighter because of the strong dependence of L on μ.
- At the same time, the effective temperature shows a monatonic decrease, as in Figure 19.1, right. This is due to the increasing radius, which increases faster than the luminosity (especially compared to lower-mass stars).
- If the core of the star grows in size due to convective overshoot, it will also extend the MS lifetime and make the star brighter.
- Figure 19.2 (right) shows the core hydrogen mass, and note the shrinking convective core of the higher-mass star over time.
- One of the main reasons for this is the reduced opacity as H is converted into He and the electron scattering processes decreases.
- Also note in the higher-mass star in Figure 19.2 that since hydrogen burning is negligible at the edge of the convective core during the main-sequence phase, the hydrogen profile established during this phase reflects the decrease in the extent of the core. In contrast, the last model is in the hydrogen shell-burning phase, the helium core has grown substantially beyond the smallest extent of the convective core.
- Figure 19.4 shows a comparison of core mass size for a $1M_\odot$ and $20M_\odot$ model in both relative and absolute visualizations.
- Note in this figure that even though the luminosity saturates at a high value very close to the center, the convective region of the core is quite extensive.
- The size of the convective core increases as the mass of the star increases too, due to the higher central temperatures.
- We can estimate the main-sequence lifetime of a star. If $\varepsilon_{\rm H}$ is the energy per unit mass per unit time of hydrogen burning, we know that

$$\tau_{\rm MS} \propto \frac{q_c \varepsilon_{\rm H} M}{L}, \tag{19.3}$$

  where q_c is some fraction of the stellar hydrogren mass that actually participates in nuclear burning.

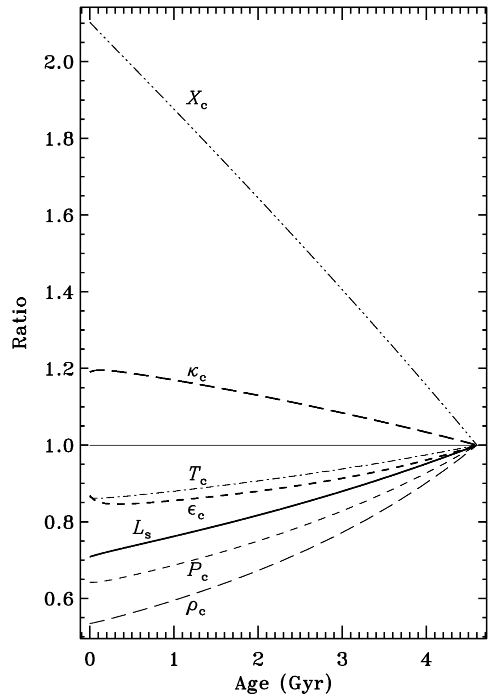

Figure 19.3: The changes in the solar properties at the center of a solar model. All variables are normalized with respect to the present Sun. From Christensen-Dalsgaard [2008].

- If $\varepsilon_{\rm H}$ and q_c are roughly independent of total stellar mass, and we assume that $L \propto M^\gamma$ as we showed before, then

$$\tau_{\rm MS} \propto M^{-(\gamma-1)}, \tag{19.4}$$

where we found $\gamma \approx 3-5$, and the relation is written to emphasize that the exponent is always negative, and main-sequence lifetime is inversely related to mass.

A note about very low mass stars

- Stars below about $0.3M_\odot$ are fully convective on the MS.
- They have large opacities due to low temperatures and very high densities.
- The densities are high because the stars need to contract to build up high enough temperatures for nuclear fusion.
- Only the PP-I chain can operate, so helium-3 is never really destroyed.
- But at the lower mass limit (high ρ), electron degeneracy kicks in.
- Conduction is very efficient here, which then cools the core below the minimum ignition temperatures.
- Some lithium or deuterium is burned, but these objects become brown dwarfs and cool down like white dwarfs.
- The difference between them and WDs is that they are fully mixed chemically, and their degenerate electrons do not move relativistically.

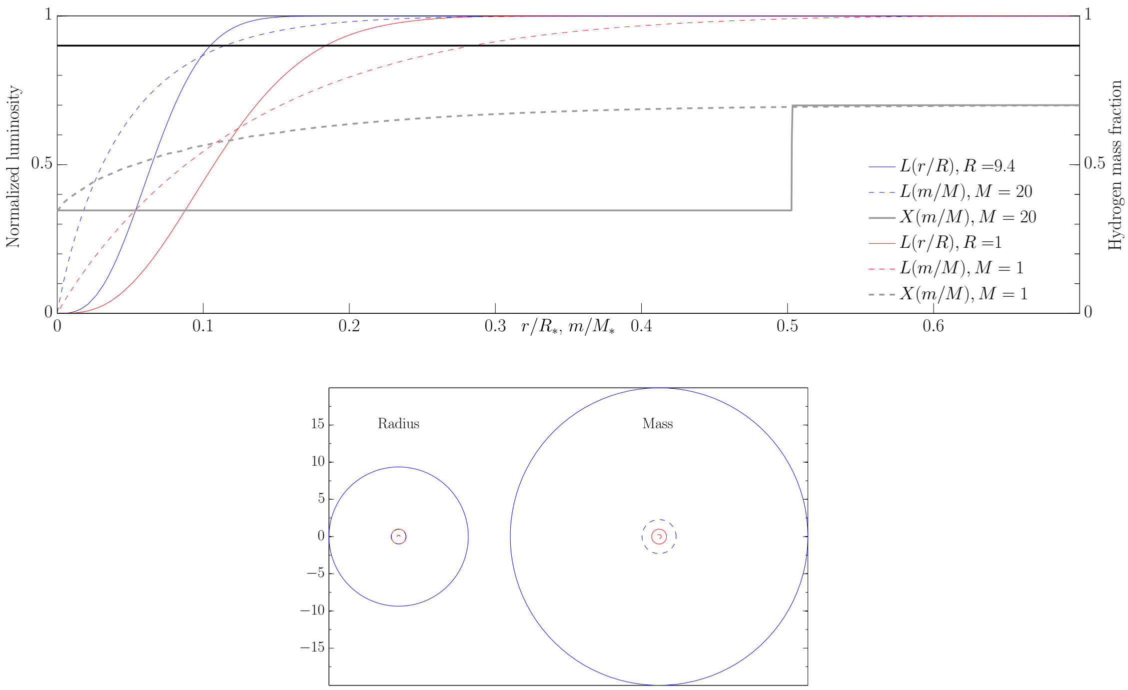

Figure 19.4: Comparison of core sizes for 2 models ($1M_\odot$ and $20M_\odot$) both with $X_c \approx 0.35$ on the main sequence. The top panel shows the normalized luminosity as a function of normalized radius and mass. A horizontal line at $0.9L_{\max}$ will be used to define the approximate core boundary. The blue lines are for the massive model, and red for the less massive one. The low-mass model has a core that is fractionally larger than the core in the high-mass model. The gray solid and dashed lines are the hydrogen mass fraction, given on the right y axis. In the bottom panel, the surface (solid lines) and core (dashed lines) boundaries are shown to scale in absolute masses and radii. On the left for the $9.4R_\odot$ model (blue), we see the core boundary in this space is almost exactly the same size of the entire low-mass star (red). In terms of mass, the core size of the massive star is larger than one solar mass.

19.3 Summary of main-sequence properties

- In general, the star regulates its nuclear burning rate to maintain hydrostatic equilibrium.
- If the rate increases for some reason, the star expands, thereby decreasing its temperature and density, reestablishing equilibrium.
- Mass rules of thumb:
  - Below about 1.3 $M_\odot$ convective envelopes, above, radiative envelopes
  - Below about 1.2 $M_\odot$ PP chain, above, CNO cycle
  - Below about 1.5 $M_\odot$ late-type stars (F, G, K, M), above, early-type (O, B, A)
- Structure rules of thumb:
  - Low-mass cores
    * PP chain is sufficient to balance gravity
    * Luminosity is not too steep, and energy flux is moderate
    * Radiation is enough to carry out the luminosity from the core
    * Core is radiative
    * Since the PP chain has a low temperature dependence, the region of burning is relatively a large mass fraction of the star, as in Figure 19.2, left.

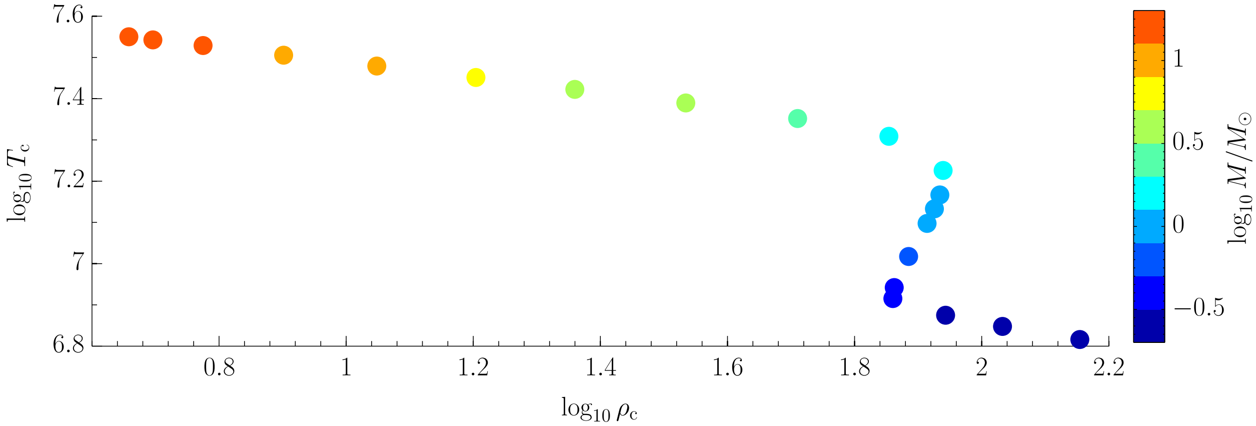

Figure 19.5: The central temperature and density for various MESA models of mass given by the colorscale. All models are just on the main sequence when $X_c = 0.68$.

- High-mass cores:
  - CNO cycle is necessary to balance gravity
  - High-temperature sensitivity ($\sim T^{20}$) means a very central energy generation region
  - Luminosity is very steep in core with a high flux
  - Temperature gradient is very steep, convection sets in
  - A convective core develops, which is very efficient
  - So efficient that here it equals the adiabatic gradient
  - Core mixing due to convection removes gradients in composition
  - The core temperature and density for different masses is shown in Figure 19.5.
  - Note the strong variations at a little above 1 solar mass stars, where convective cores start to appear.
- Low-mass envelopes:
  - Opacity is rather large because of hydrogen and helium ionization zones and corresponding bound-free transitions
  - Convection is needed to carry the radiative flux through the region; steep temperature gradient
  - Below about $0.3\,M_\odot$ the entire star is convective
- High-mass envelopes:
  - Hydrogen and helium ionized so rather low opacities
  - The radiative flux is carried out by radiation. Radiative envelope.
  - In very massive stars $> 10\,M_\odot$, some opacity peaks due to ionized iron and nickel can cause thin convection zones near the surface

- MS location rules of thumb (as we saw in homology relations):
  - Higher He content results in more luminous and hotter MS tracks.
  - The MS lifetime decreases with increasing He content.
  - Higher metallicity makes the star cooler due to increased opacity.
  - Alpha elements (O, Ne, Mg, Si, S, Ca, Ti, ...) that are enhanced in metal-poor stars produce fainter, cooler MS tracks.
  - Changing the mixing length, or convective efficiency, affects the MS.
  - There is no effect on the luminosity, but an increased effeciency sets up a lower thermal gradient.

  - This increases the effective temperature, and therefore the radius decreases.

- Evolution (on main sequence) rules of thumb:
  - ZAMS to TAMS lifetimes are much shorter for high-mass stars
  - Tracks on HR diagram are vertical for low-mass, and diagonal for high-mass stars
  - Luminosity increases due to increase in molecular weight in core
  - Low-mass stars have abundance changes in core that is smooth
  - High-mass stars have discontinuous changes due to convective mixing (although core shrinking can smooth out the discontinuities)
  - Below about $0.3\,M_\odot$ the core is completely mixed and stars live long
  - Near the end of main-sequence evolution:
    * Nearly isothermal core with zero luminosity
    * Core is very hot from high μ
    * At core boundary temperature is high enough for shell burning to occur
    * The high T and large volume of the burning region leads to high shell L
    * No thermal equilibrium, envelope expands

Unit 20

The Terminal-Age Main Sequence and Subgiant Branch

20.1 Description of evolution movies

- From here on you can use the evolution movies to study the various stages that will be discussed.
- There are 2 stars considered of different masses, $M_1 = 1\ M_\odot$ and $M_7 = 7\ M_\odot$.
- Each star has 2 associated movies, one of the time evolution of general properties, and one depicting the time evolution of interior profiles.
- They will be referred to as 1Ma, 1Mb, 7Ma, 7Mb, where "a" and "b" denote the history or profile movie, respectively.
- A particular model in a movie will be denoted by, e.g., 1Mb-200, so the 200th model of movie 1Mb (not megabyte!)

20.2 TAMS

- The terminal-age main sequence (TAMS) is roughly the stage when the all the hydrogen is exhausted from the very center of the star (not necessarily though throughout the whole core).
- This is about at model 280 for the 1 $M_\odot$ star and model 1200 for the 7 $M_\odot$ star.
- Observationally, we can only really talk about stars greater than about $0.8M_\odot$, since less massive stars are still on the main sequence (of course, we can use theory to talk about lower-mass stars).
- To summarize the previous material, as stars evolve on the main sequence they go **above** the ZAMS up and to the right or left depending on mass.
- Notice that this is only the case for chemically inhomogeneous models (if a star remained mixed and the mean molecular weight increased with time throughout, it would evolve below the ZAMS for a given mass, as we saw in our homology relations earlier).
- When the central hydrogen content reaches about $X_c \approx 0.05$ (see points 3 in Figure 20.1 and 2 in Figure 20.2) for stars above about $1.1M_\odot$, the opacity is dropping (increased He), and the envelope luminosity is greater than the energy generation in the core (not much H left!)

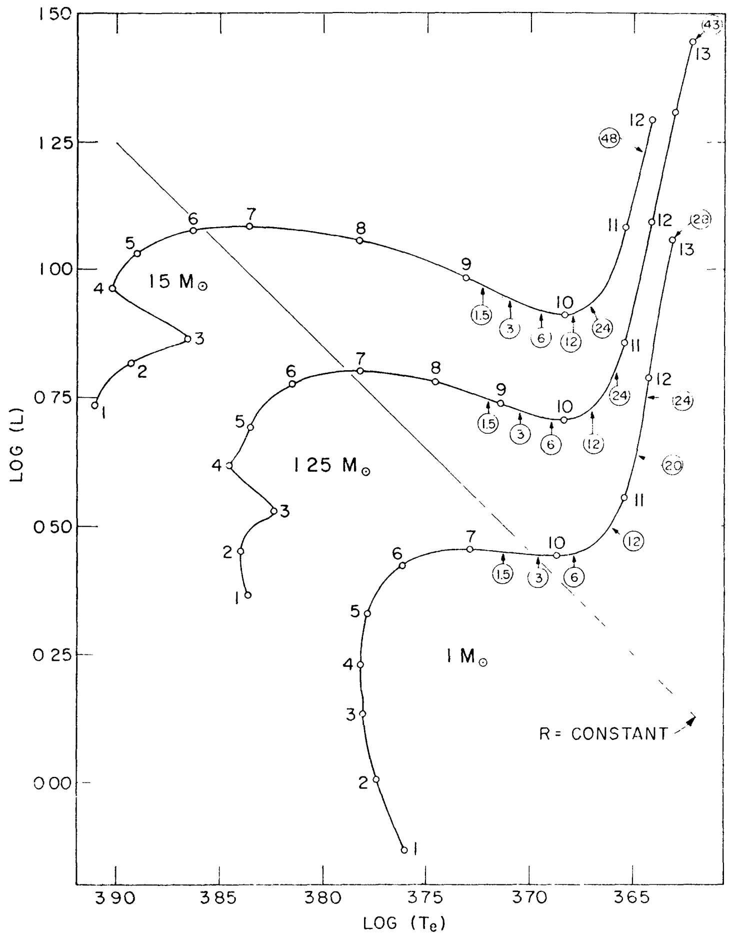

Figure 20.1: Evolutionary tracks for low-mass Pop. I stars. Basically, points 1-3 are the ZAMS to TAMS. From Iben [1967b].

- The star shrinks on a Kelvin-Helmholtz time scale to make up for the excess luminosity, then the effective temperature increases a bit (see § 20.3). This is a called *overall contraction.*
- This causes the little wiggle (or "hook") on the HR diagram. Low-mass stars do not show this because they do not need to contract so much because the luminosity was never that great. See Figure 20.1 (points 3 to 4) and 7Ma-1200-1220.
- The main difference is the higher-mass stars have a convective core.
- The higher-mass cores deplete H over larger regions, and thus the contraction is more drastic as to maintain nuclear burning at the right level.
- Nonetheless, near the TAMS as $X_c \to 0$ for all masses:
  - Core is mainly filled with inert helium (too cool to burn, needs 10^8K)
  - But there is a large T_c and μ
  - Core is isothermal since $\varepsilon \to 0$ and then $\mathrm{d}T/\mathrm{d}r \to 0$ (see Equation (13.19)).
  - The temperature at the core boundary is high enough, however, to ignite leftover hydrogen
  - The contraction has pulled in H to hotter and denser regions (still the shell), so the shell ignites!
  - The shell burns H and adds He to the core, whose mass increases and it contracts more, heating it up (eventually to ignite He later on)

- All of this emphasizes the **Shell-burning law**: When a region within a burning shell contracts, the region outside the shell expands; when the region inside the shell expands, the region outside the shell contracts. We will see this behavior again and again.
- Despite many efforts, and the fact that numerical experiments show that this law is true, it is not obviously clear why it is the case.

20.3 Schönberg-Chandrasekhar Limit

- Let's look at what's happening in the core at the TAMS. Can it support the growing mass in the overlying layers from outer core burning?
- In 1942 Chandrasekhar and Schönberg studied hydrostatic equilibrium for an isothermal He core and an ideal equation of state.
- Assume constant core temperature, and that the envelope provides a pressure $P_{\rm env}$. The goal is to compute the core pressure P_c.
- Consider hydrostatic equilibrium and multiply both sides by $4\pi r^3$ and integrate in core (recall Equation (B.6)):

$$\int_0^{R_c} 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = -\int_0^{R_c} \rho \frac{Gm}{r^2} 4\pi r^3 \mathrm{d}r = E_{g,\rm c} \tag{20.1}$$

- Integrate by parts and use ideal gas law

$$4\pi R_c^3 P_c - 3\frac{M_c k_{\rm B} T_c}{\mu m_{\rm u}} = E_{g,\rm c}. \tag{20.2}$$

- If we assume that the density is the mean core density $\rho \approx 3M_c/4\pi R_c^3$, then

$$E_{\rm g,c} \approx -\frac{3}{5}\frac{GM_c^2}{R_c}. \tag{20.3}$$

- Solving everything for P_c, we get

$$P_c = \frac{3}{4\pi R_c^3}\left(\frac{M_c k_{\rm B} T_c}{\mu m_{\rm u}} - \frac{1}{5}\frac{GM_c^2}{R_c}\right) \tag{20.4}$$

- The core pressure must match the envelope pressure for equilibrium, and must adjust its radius to do so.
- Can it always do so? Its maximum value is when

$$R_c = \frac{4}{15}\frac{GM_c \mu m_{\rm u}}{k_{\rm B} T_c}, \tag{20.5}$$

which gives

$$P_{\rm c,max} = \frac{10125}{1024 G^3 M_c^2}\left(\frac{k_{\rm B} T_c}{\mu_c m_{\rm u}}\right)^4. \tag{20.6}$$

- As you can see, as the core mass increases, the core pressure will drop and at some point may fall below the envelope pressure.
- The mass at which this happens is the Schönberg-Chandrasekhar limit.
- We know from hydrostatic equilibrium that $P_{\rm env} \propto M^2/R^4$.

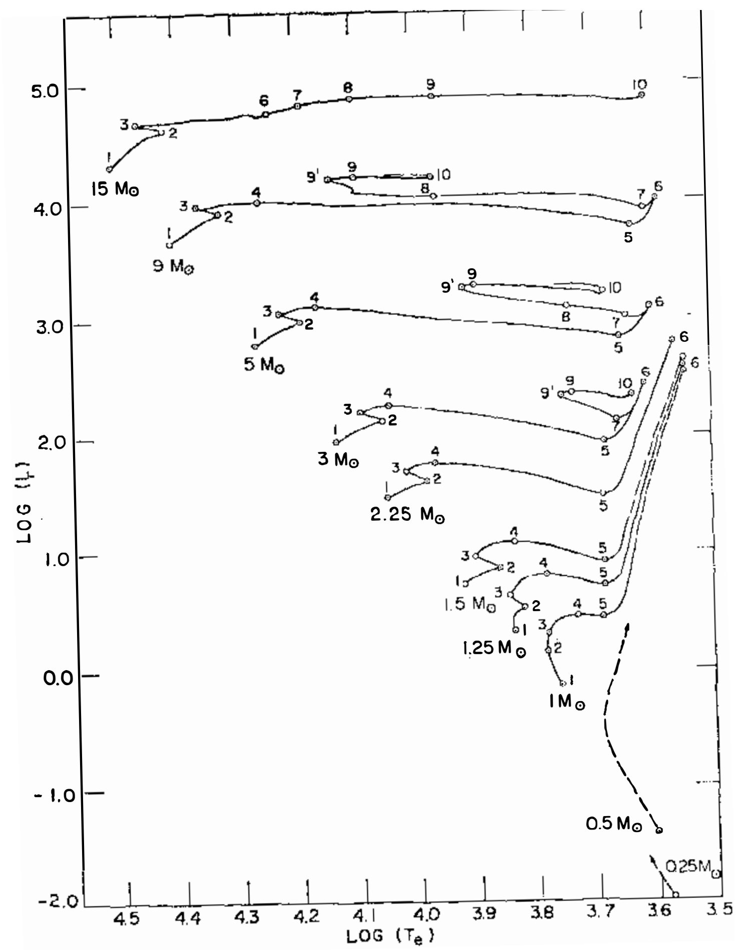

Figure 20.2: Paths in the HR diagram for a range of masses and solar metallicity. From Iben [1967a].

- From homology, we can find that $P_{\rm env} \propto T_c^4/M^2$
- So the pressure at the surface of the core is independent of the core size.
- Using the right coefficients, it is then easy to show that

$$\frac{M_c}{M} \approx 0.37 \left(\frac{\mu_{\rm env}}{\mu_c}\right)^2. \quad (20.7)$$

- If $\mu_{\rm env} = 0.6$ and $\mu_c \approx 1$ (at solar composition), then the limit is roughly

$$\frac{M_c}{M} \approx 0.13. \quad (20.8)$$

- Above this limit, which will likely occur for stars greater than $3M_\odot$:
  - the isothermal core contracts rapidly
  - the density increases, the temperature increases and nuclear reactions speed up in the shell
  - This pushes in both directions and mass is lost in the shell, and burning is in a thin shell
  - Even though the energy rate increases, the luminosity decreases a bit because of the mass loss in the shell

  - Since the timescale is faster than the nuclear one, the stars become redder very quickly
  - This leads to observational Hertzsprung gap (points 4 to 5 in Figure 20.2)
  - Not many stars have time to be "observed"
  - These are subgiant stars on the way to the bottom of the red-giant branch
  - After the wiggle for our 7 $M_\odot$ model, the core He mass is about 1 $M_\odot$, above the limit computed above.

- For low-mass stars ($\leq 1.3M_\odot$), the helium core is somewhat degenerate and higher pressures are present, so this limit is not applicable and the approach to the RGB is slower.
- For higher-mass stars, the contraction happens very quickly and isothermal cores never actually have time to set in.

20.4 The subgiant branch

- To summarize the above once again, in general, the move across the H-R diagram to the right defines the subgiant branch (SGB).
- These are models 300-330 for the 1 $M_\odot$ star, and 1200-1500 for the 7 $M_\odot$ star.
- In general, cores are shrinking, envelopes are expanding, and surface temperatures are being reduced.
- Higher-mass stars have He core masses above the C-S limit.
- The envelope is adjusting to a new H-burning shell.
- The luminosity is larger as the burning takes place at a higher temperature than it was in the core
- With a large luminosity the shell has a difficult time radiating it (it will eventually become convective)
- But right now it absorbs the luminosity, heats up, and expands
- The Virial theorem shows some of the energy goes into expansion, not all of it makes it to the surface
- The slope of the luminosity in this move across the HR diagram depends on mass
- This stage should happen over the timescale of shell burning, a nuclear timescale ...
- But other influences may affect it, as will be discussed

Unit 21

Towards and up the Red-Giant Branch

21.1 High-mass stars

- The red-giant branch (RGB) for the 7 $M_\odot$ star is approximately models 1500-1750.
- As the star approaches the bottom of the RGB, opacities increase because of the cooling outer layers, and a convection zone near the surface appears. See 7Ma-1500 in the "Kippenhahn" panel.
- Mixing occurs near the surface.
- H^- is the dominant opacity source.
- As the effective temperature continues to decrease, the luminosity finally begins to increase as convection allows the luminosity to escape more readily.
- The shell burning power increases, but at the same time the mass in the shell decreases.
- The variation in the luminosity is interesting, as you can see that most of it is generated in a thin shell at about $m \approx 1.1\ M_\odot$ in 7Mb-1530 in the "Summary Profile" panel.
- But note that near the center there is a small amount due to initial He burning that is beginning to take place. See the "Power" panel and the "text" panel.
- The subtle decrease in L in the envelope is due to expansion of the outer envelope where no nuclear sources reside and gravitational energy is "lost."
- Note the core is no longer isothermal as a temperature gradient is required to carry out the luminosity supplied by He burning ("Summary Burn" panel).
- Continuing on, the convective envelope increases its extent in depth, carrying the luminosity to the surface, as the core keeps contracting and the star keeps getting bigger.
- The convection almost reaches the burning shell at model 1600, but mixes the star and creates interesting abundance ratios for observers.
- A small convective core sets in with N being converted to O (recall one of the intermediate reactions in the CNO cycle).
- When around model 1800 is reached, the central temperature and density are such that He burning via the triple-alpha process takes place and is steady.

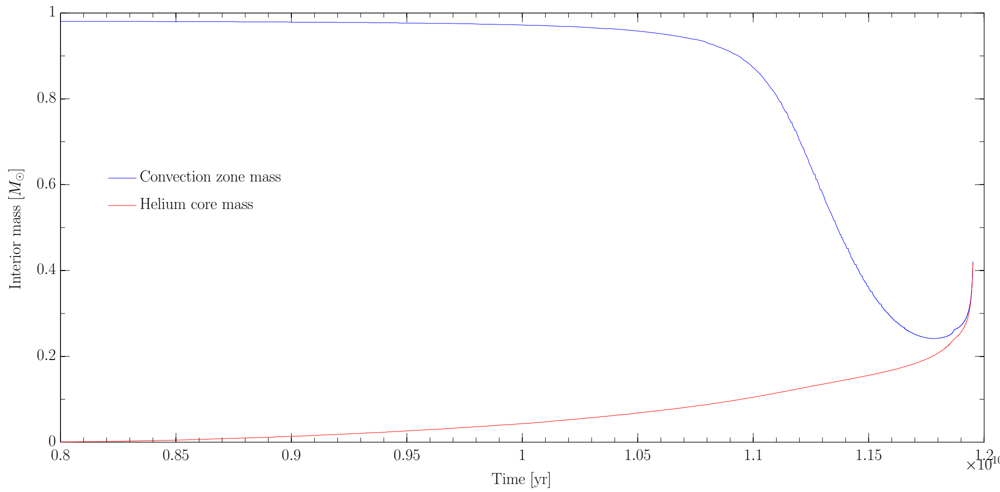

Figure 21.1: How the base of the convection zone and the mass of the He core change with time for a $1M_\odot$ model. This illustrates how the first dredge up occurs.

- This occurs at about $T = 10^8$K.
- Now, instead of gravitational contraction in the core, nuclear energy again can support the star.
- The star "leaves" the RGB.
- It is a rapid increase in enery such that it causes the core to expland (energy can't get carried out fast enough).
- There is still shell burning that is supplying most of the stellar luminosity, but it slows down a bit, and the luminosity decreases abruptly.
- Because of the shell-burning law too, the expanding core causes a shrinking star.

21.1.1 Low-mass stars

- The RGB phase for the low-mass star is models 330 to 4500. There are many more short time steps in this case because of the degenerate condition in the core.
- The He core stars below the C-S limit, unlike the high-mass case.
- The density is also higher and so there is a degenerate component.
- The degeneracy provides enough pressure so that core contraction is not as extreme as for higher-mass stars.
- The strong degeneracy is clear in the "temperature-density" panel.
- A convective envelope develops down to about $0.2M$, and results in a sharp change in $X_{\rm H}$ ("summary profile" panel).
- The convection mixes material from the surface to a point that once produced He from H. When it reaches its deepest extent, this is the *first dredge up*. What happens?
  - The surface He abundance increases, and ^{3}He and CNO elements get mixed in.
  - A doubling of the surface ^{14}N abundance.
  - A reduction of ^{12}C by about 30%.

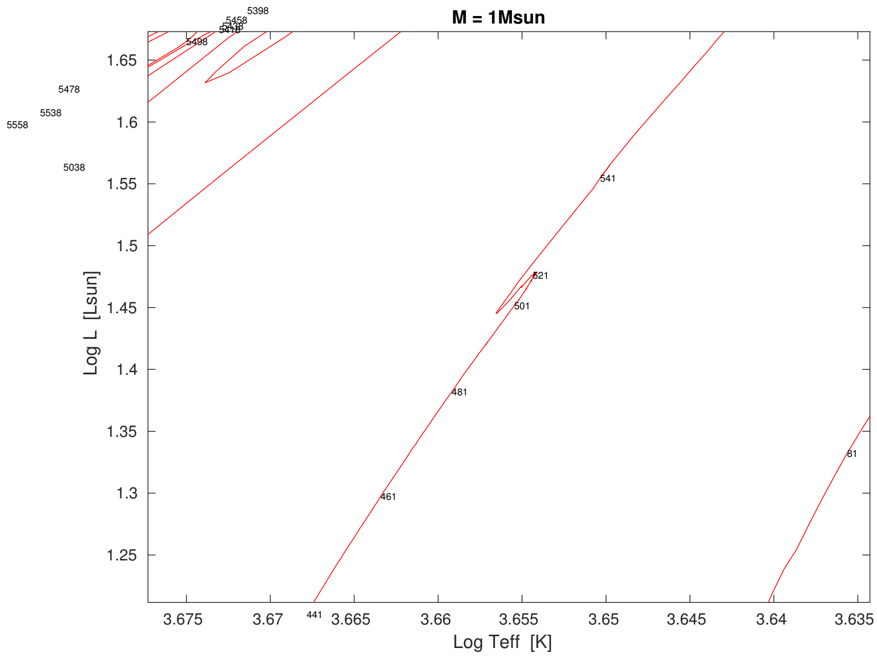

Figure 21.2: A zoom in on the H-R track of the $1\ M_\odot$ model during the RGB luminosity bump. The model numbers are annotated.

  - Formation of $^{12}C/^{13}C$ of about 20-30.
  - A reduction of surface lithium and berrylium abundances by a few orders of magnitude.
  - Some of these can be seen in the "abundance" panel as the deep mixing occurs.

- See Figure 21.1 for an illustration.
- Because of degeneracy, the core is dominated by heating by contraction, since the thermal energy of degenerate electrons is independent of temperature.
- The H-burning shell continues to move outwards into fresh hydrogen layers.
- The convection zone retreats, leaving behind a chemical discontinuity.
- When the shell reaches this area, the RGB motion briefly goes down due to a decrease in H burning because of the lower mean molecular weight (see 1Ma-515 in the "H-R" panel)
- After it crosses the discontinuity, the mean molecular weight is again constant and the luminosity increases again.
- Therefore, the star crosses the same luminosity point 3 times, increasing star counts at this level.
- This is the *RGB luminosity bump*, and represents about 20% of the RGB lifetime.
- A zoom of this is shown in Figure 21.2.
- Along the RGB, the core is becoming denser and denser, and some gravitational energy is being produced.
- However, there are some substantial energy losses due to neutrino production, and sometimes these can be greater than the gravitational energy release.
- There are potentially 3 neutrino production mechanisms that don't involve nuclear processes:

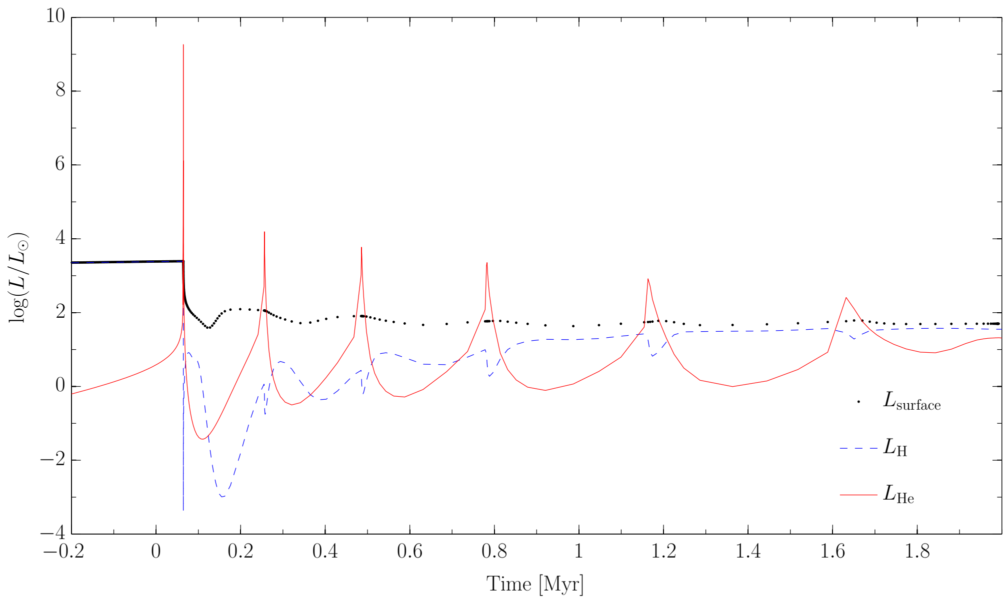

Figure 21.3: Helium flash for a $1M_\odot$ model. Time has been shifted to approximately the start of the flash, which corresponded to about 12 billion years. Note the decrease in the surface luminosity due to the expansion of the He core and cooling of H-burning shell.

- Pair annihilation processes are when high-energy photons produce electron-positron pairs, that annihilate and produce photons again, or, rarely, a neutrino-anti neutrino pair. This typically only happens above one billion degrees.
- Photoneutrino processes from Compton scattering (electron + photon), when the photon becomes a neutrino-anti neutrino pair.
- Plasma processes when the photon traveling in a dense, degenerate environment becomes like a plasmon (having mass) and decays into a neutrino-anti neutrino pair.

- This can lead to the $\mathrm{d}L/\mathrm{d}m = \varepsilon < 0$ in the innermost regions.
- The maximum temperature is now located off center, in a shell. See 1Mb-3000 in the “summary burn” panel.
- Another cooling process is due to conduction from the degnerate electrons.
- In any case, the maximum temperature, wherever it is located, is increasing as time goes on because of the increasing He core mass and gravitational heating up of the inner layers.

21.2 Helium flash

- The He flash occurs after model 4000 or so.
- The temperature of the burning shell is increasing up the RGB, and when it reaches 10^8K, the *helium flash* occurs.
- It coincides when the He-core mass is about 0.5 $M_\odot$ (about 0.46 in our case), mostly independent of the total stellar mass.
- In a more massive star (which would be non-degenerate matter), the release of energy by nucelar burning would cause an increase in pressure and an expansion of material (core), and then cooling and an equilibrium would be restored, all rather smoothly.

- In the degenerate case, however, the gas pressure is basically independent of temperature (recall the EOS), and the high temperature causes no immediate reaction at the core.
- Instead the nuclear energy generation increases → higher temperatures → increased energy generation, ...
- This is called a thermal runaway.
- The local luminosity in the core increases to about 100 billion $L_\odot$ in a few hours! 1Ma-4300 in the "kippenhahn" panel shows the luminosity increase.
- See Figure 21.3.
- The new luminosity does not make it to the surface, however, but is absorbed by the overlying layers, which expand just outside the H-burning shell.
- Convection also sets in which spreads out the energy production over more mass layers.
- Eventually, the temperature gets so high that degeneracy is "lifted" at the point where the flash occurs (density is roughly constant). Recall Equation (8.8):

$$\frac{\rho}{\mu_\mathrm{e}} > 2.4 \times 10^{-8} T^{3/2}.$$

- Interior to this first outer layer explosion, some smaller flashes may take place which eventually removes the degeneracy everywhere. This is seen well around 1Ma-5650 in the "temperature-density" panel, where the degeneracy is removed.
- After this, the core expands (envelope contracts!) and cools, and an equilibrium of helium burning in the core proceeds.
- The dynamical time scale of the star, because it is so large, is of the order of months.
- So the He flash in the core is not visible at the surface. The whole process takes on the order of one million years.
- See Figure 21.3 again.
- It is now on the *helium burning main sequence*, or better, the *horizontal branch*, or for seismologists, the *red clump*.

Unit 22

Red-Giant Branch Morphology

22.0.1 RGB properties

Here we discuss the main RGB features on various physical and chemical parameters. The main morphological features are it's location on the H-R diagram, the luminosity when the bump occurs, and the luminosity at the tip of the RGB.

The main drivers of these features are the star's mass, it's He core mass, the composition (Y, Z), and convective efficiency.

22.1 RGB location

- The main determinator of the RGB location is the size of the convective envelope.
- With decreasing mass, the RGB is cooler.
- An increase in He content reduces the opacity, causing a shrinking of the convective envelope, and thus a hotter RGB.
- An increase in metallicity produces a deeper convection zone (higher opacity, cooler temps) and a cooler RGB.
- An increase in the convective efficiency, such as an increase in the mixing length, the RGB shifts to hotter effective temperatures.
- If the mixing length parameter is set to zero, the RGB disappears and expands until it falls apart.

22.2 RGB bump luminosity

- This phenomenon depends most strongly on the location of the H-abundance discontinuity after the first dredge up.
- The bump luminosity decreases as this location moves deeper into the star, as it will encounter it at earlier times.
- A decrease in He, or increase in metals, pushes this location deeper, and reduces the bump luminosity.
- More efficient convection, decreases the mass extent of the outer convection zone and the bump occurs at higher luminosity.

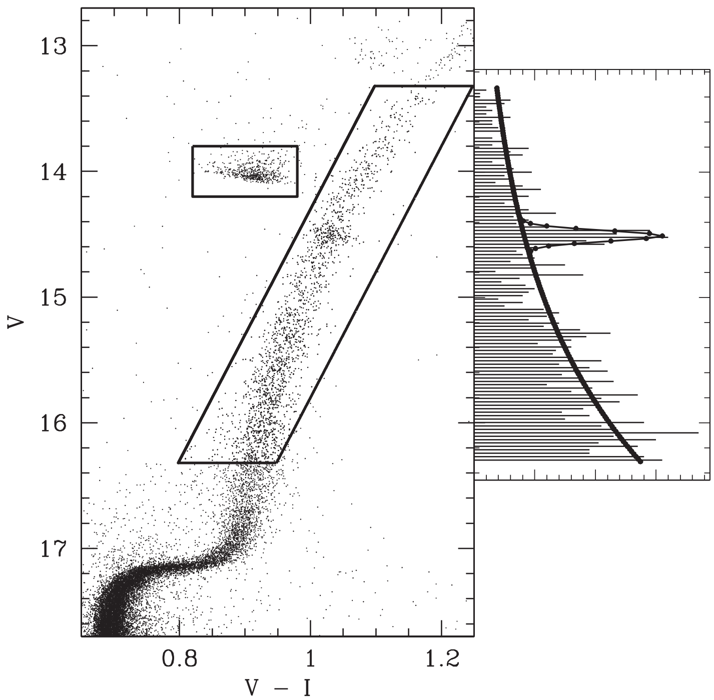

Figure 22.1: Left panel: the color–magnitude diagram of HST data for the globular cluster 47 Tuc. The RGB (including the RGBB) and the HB are all contained within their respective color–magnitude selection boxes. Right panel: magnitude distribution of RG stars. The RGBB stands out as a prominent and significant peak at $V = 14.51$, with a normalization of (122 ± 14) stars. From Nataf et al. [2011], where they show the lifetime of the RGBB is different for different He amounts in the cluster stars.

- Another prediction of stellar evolution theory is that the *lifetime* of the RGBB is decreased as the He content increases.
- Empirical support for this is alluded to in Figure 22.1 and the associated article.

22.3 RGB tip luminosity

- The luminosity at the tip of the RGB occurs when He is ignited.
- This typically happens at a well defined He core mass.
- For stars less massive than about $1.8M_{\odot}$, the mass of the He core at the flash does not depend on the overall mass that much, and they all develop about the same amount of electron degeneracy in the core.
- So the luminosity at the flash is about the same for these stars (all things otherwise being equal).
- For higher masses (about less than $3M_{\odot}$), the mass of the core is smaller and degeneracy is at lower levels, so the luminosity is reduced at the tip (ignition occurs earlier).
- For higher masses still, the luminosity starts to increase again as a result of the mass of the He core increasing again.
- An increasing He content increases interior temperatures and decreases electron degeneracy leading to lower He core mass and a lower tip luminosity.

- An increasing metallicity also helps lower the He core mass, because shell H burning is more efficient.
- This heats the He core faster; however, the luminosity is higher with increasing metals since L is strongly affected by the H-burning shell.
- Convection changes do not affect the tip luminosity since these don't really change the mass of the He core.

Unit 23

The Horizontal Branch

23.1 Quick tour of non-hydrogen nuclear reactions

- After H burning, we have He burning through the general triple alpha process
- There are, however, no stable elements with $A = 5$ or $A = 8$ (lithium is 7 and berrylium is 9).
- So it's rather tricky for helium to burn (it can't just interact with hydrogen or with itself).
- There is however an isotope of berrylium present from its formation from 2 helium nuclei

$$^4\text{He} + ^4\text{He} \longrightarrow {}^8\text{Be} \tag{23.1}$$
$$^8\text{Be} + ^4\text{He} \longrightarrow {}^{12}\text{C} + \gamma. \tag{23.2}$$

- The ^{8}Be ground state energy is about 100keV higher than the ground state of 2 He nuclei, so it wants to decay into that, to find the lowest ground state.
- Its lifetime is only about 10^{-16}s.
- Since ^{8}Be decays so quickly, the third helium must arrive in a short amount of time.
- But this is orders of magnitude longer than a scattering event.
- Additionally, at high temperatures the $\alpha + \alpha$ reactions increase rapidly.
- The key is that a nucleus of carbon is produced before the berrylium decay.
- Energy release is about 7.3MeV, or about 0.6MeV per nucleon, which is about an order of magnitude smaller than CNO H burning.
- This all takes place in the $T = 1 - 2 \times 10^8$K range, and the reactions can be written

$$\varepsilon \approx \varepsilon_0 Y^3 \rho^2 T^\nu, \tag{23.3}$$

  where $\nu = -3 + 4.4/T_9$.
- So at $T = 10^8$, $\nu \approx 40$!
- As density increases, this reaction is favored due to the quadratic dependence in the rate.

- After a supply of carbon is produced, there are successive α particle captures to produce some "alpha elements"

$$
\begin{aligned}
{}^{12}\mathrm{C} + {}^{4}\mathrm{He} &\longrightarrow {}^{16}\mathrm{O} + \gamma \\
{}^{16}\mathrm{O} + {}^{4}\mathrm{He} &\longrightarrow {}^{20}\mathrm{Ne} + \gamma \\
{}^{20}\mathrm{Ne} + {}^{4}\mathrm{He} &\longrightarrow {}^{24}\mathrm{Mg} + \gamma \\
{}^{24}\mathrm{Mg} + {}^{4}\mathrm{He} &\longrightarrow {}^{28}\mathrm{Si} + \gamma
\end{aligned}
$$

- The conversion of helium and carbon into oxygen is extremely important, as the C/O ratio is critical for understanding carbon-oxygen white dwarfs and their cooling times.
- And when the amount of He begins to be reduced in the core, this reaction competes with the main 3α reaction, and affects the He-burning lifetime overall.
- In 7Mb-2095 or so, the "abundance" and "burn" panels show this.
- The nuclear cross section is not that precisely known, however, due to a resonance and a very small value at low energies, making it difficult to measure.
- Note that the isotopes with an atomic weight in multiples of 4 are known as the *alpha* elements.

23.2 The horizontal branch properties

- Stars with $M \leq 2.3 M_{\odot}$ develop degenerate He core, go through helium flash.
- Stars with $2.3 \leq M/M_{\odot} \leq 9$ just start burning He, but will NOT ignite carbon, later (see Figure **??**).
- For most stars, He is now burning in the core (non-degenerately) and there is still an H-burning shell.
- When this is happening "quietly," the star is on the zero-age horizontal branch (ZAHB).
- The core is convective due to the large luminosity associated with He burning (Fig. **??**).
- Models predict a horizontal distribution on the HR diagram of these stars.
- The efficiency of the H-burning shell is modulated by the mass of the overlying envelope.
- The more massive, the hotter the burning.
- The effective temperature of these stars depends on mass of the envelope.
- The more massive envelope, the cooler (from inertia).
- Stars can fall into the "Red Clump" on the horizontal branch, which have large envelope masses.
- Less-massive enveloped stars are bluer.
- Also, the horizontal branch is not exactly horizontal, as more massive stars are slightly brighter.
- The luminosity is fixed mostly by the mass of the He core, and then by the mass of its envelope.
- Since the He core mass is almost constant for low-mass stars, the horizontal branch luminosity is an important distance indicator.
- Where cluster stars fall here can be a tricky problem, as metallicity and mass loss play a role in all of this.
- For increased He content, the blue part of the ZAHB (lower mass stars) becomes fainter and the red part brighter.

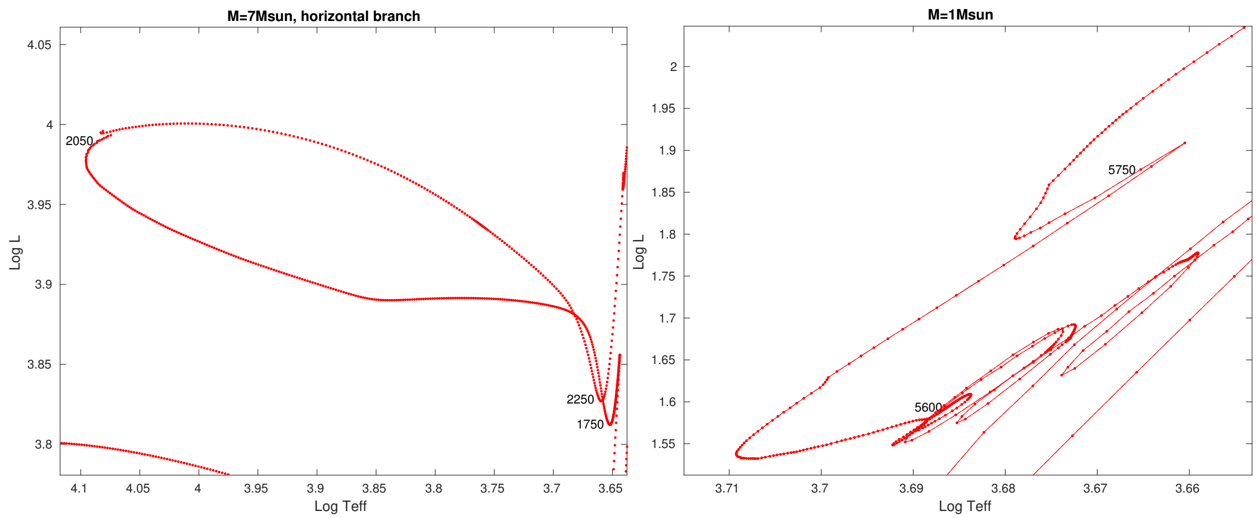

Figure 23.1: Zoom in on the horizontal branch for the 7 $M_\odot$ models (left) and 1 $M_\odot$ models (right). The model numbers are annotated.

- An increase in metals makes the ZAHB fainter and cooler, due to the lower core He mass at the flash and the increased opacity.
- Any process(es) leading to mass loss along the RGB, delaying the He flash, will lead to a hotter and brighter ZAHB.

23.3 Horizontal branch evolution

23.3.1 High-mass stars

The $7M_\odot$ star is on the horizontal branch in models 1750 to 2250, approximately. See Figure 23.1 for a zoom of the horizontal branch.

There are a few defining characteristics of the horizontal branch that need to be pointed out:

1. The (convective) core is burning He and a shell is burning H. Central He burning lasts about $1-2\times10^7$ yr.
2. The surface convection zone from the RGB is gone (see "kippenhahn" panel)
3. The high-mass star goes through a striking loop:
   - The star first goes blueward until model 2050, then redward on the H-R diagram.
   - In the blueward direction, the H-burning shell maintains an even level of efficiency and the He-burning core increases.
   - More than half of the luminosity still comes from shell burning.
   - As He burning briefly gets stronger, the core is expanding and the envelope shrinking.
   - At model 2050, the He in the core is $Y \leq 0.5$, and $L_{3\alpha} \approx 20\%$ of the total luminosity.
   - When the core starts to decrease in luminosity as He is running low, the star moves redward.
   - Figure 23.2 depicts these features.
4. The number of loops, and how far "blue" they go, depends on stellar mass.

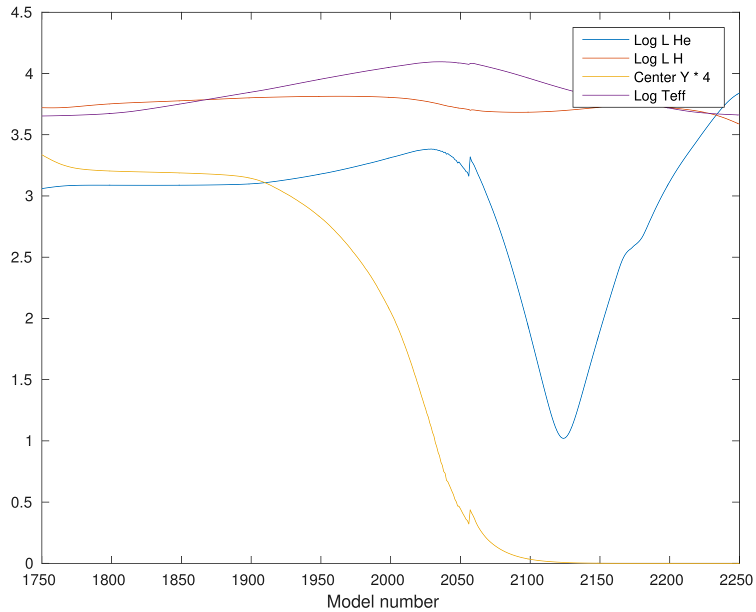

Figure 23.2: The luminosities, effective temperature, and core He content in the massive star along the horizontal branch. The He mass fraction has been multiplied by 4 to scale it near the other quantities.

- The more massive a star, the longer the loop to the blue.
- That's because the H-burning shell contributes a significantly greater amount of energy in intermediate-mass stars than for low-mass stars.
- Lower-mass stars don't have significant loops.
- Increasing He abundance extends the blue loop, as does lowering the metallicity. However, models show nonlinear behaviors here.
- Increasing core convective efficiency reduces the extension of the blue loop.
- These loops are on nuclear burning timescales.
- See Figure 23.3.

5. Also note the *Instability Strip* crosses the horizontal branch, where stars pulsate in long periods (RR Lyrae stars, Cepheids, more later).

23.3.2 Low-mass stars

The 1 $M_\odot$ star is on the horizontal branch in models 5600-5750. See Figure 23.1. There isn't too much more to add beyond the high-mass discussion:

1. There is no significant blue loop for lower-mass stars.
2. The He-core burning produces a carbon-oxygen core in about a 30/70 ratio when He is fully depleted. Figure 23.4 shows this.
3. This core is degenerate, as the "temperature-density" panel shows around 1Ma-5795.

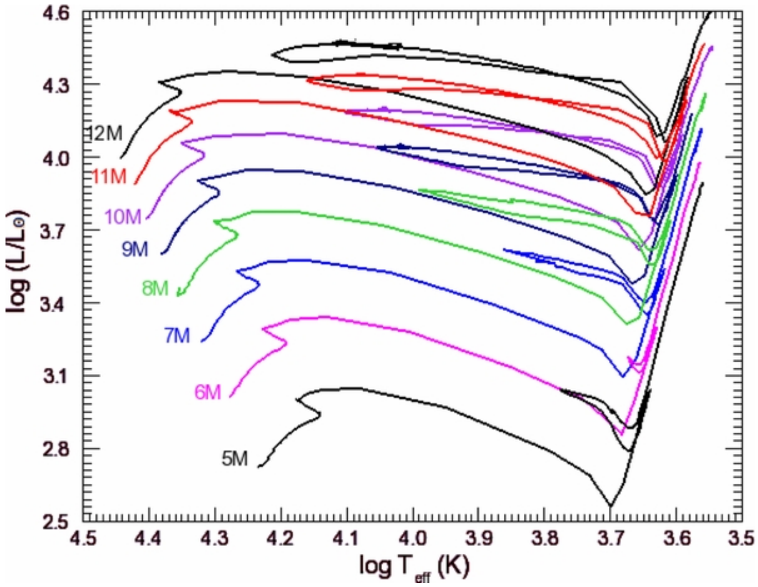

Figure 23.3: Evolutionary tracks for intermediate-mass stars and higher, showing the extent of the blue loops.

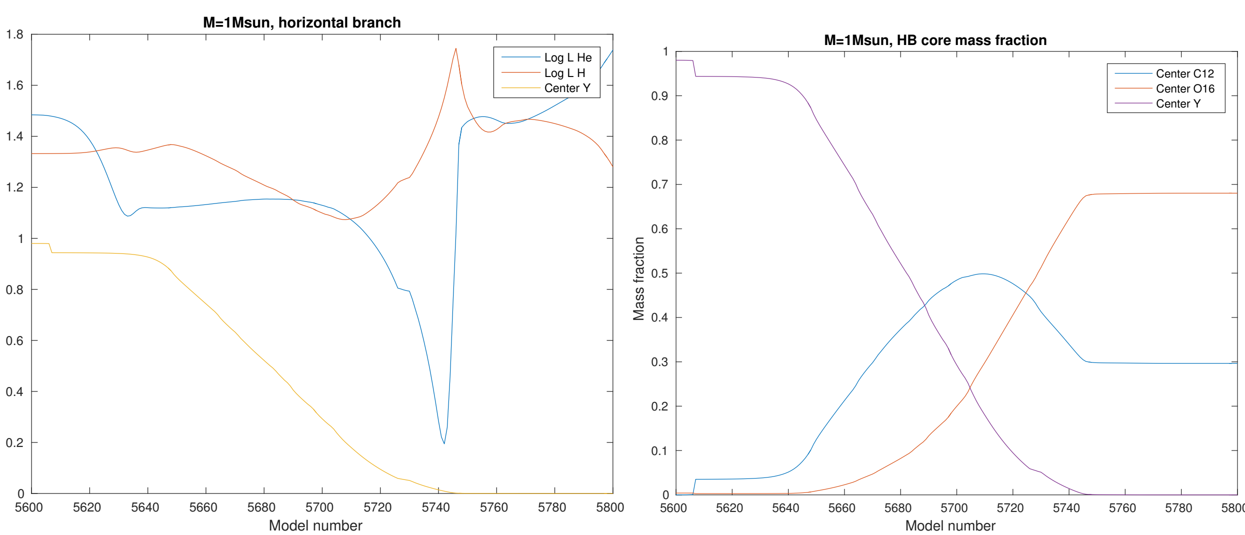

Figure 23.4: Horizontal branch properties for the $1\ M_\odot$ star.

Unit 24

Asymptotic Giant Branch

24.1 General overview

- The models representing the AGB for the 1 $M_\odot$ star are approximately 5750-6000.
- The models representing the AGB for the 7 $M_\odot$ star are approximately 2260-2495, although the full AGB evolution was not computed.
- The H-R tracks are shown in Figure 24.1
- To summarize the approach to the AGB after horizontal branch evolution, the main driver is that helium is less and less available for fusion in the core.
- As C and O builds up in the core, the mean molecular weight increases.
- The core contracts and increases in temperature (as before, in the Hertzsprung Gap).
- The contracting core releases gravitational energy and some gets converted to thermal energy and it reignites He in a shell around the core.
- The shell-burning law kicks in and the envelope expands, star moves to the red.
- There are 3 main characteristics of the AGB:
  1. Nuclear burning takes place in 2 shells (with an He layer in between). As He burning in the core exhausts, a new shell of He burning takes over in addition to the H-burning shell.
  2. The luminosity is determined by the core C-O mass only.
  3. A strong stellar wind due to radiation pressure develops causing significant mass loss.
  4. *s*-process elements are produced.
- Let's look at some of these.

24.2 Double-shell burning

- Some important interior profiles of the 7 $M_\odot$ star on the AGB during double-shell burning are shown in Figure 24.2.
- The 2 burning shells are evident from the $\varepsilon_{\rm nuc}$.

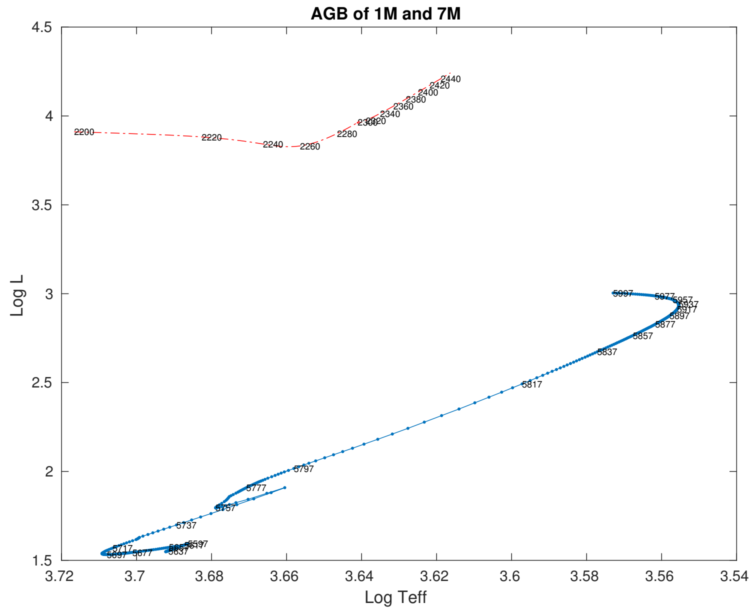

Figure 24.1: Zoom of the asymptotic giant branch of the 1 $M_{\odot}$ star (blue) and 7 $M_{\odot}$ star (red).

- The luminosity is interesting and notable: Remember that $\mathrm{d}L/\mathrm{d}m = 0$ unless energy is being generated, so here we'd really have to account for $\varepsilon_{\rm nuc} + \varepsilon_{\rm grav}$.
- The finite luminosity at the innermost region is from gravitational contraction of the core, as $\varepsilon_{\rm grav} > 0$.
- Between the 2 shells, the luminosity is decreasing slightly as that region expands (does work against gravity), and $\varepsilon_{\rm grav} < 0$.
- In the outer part above the H shell, the envelope is also contracting, releasing gravitational energy.
- This is the shell-burning law in triple!

24.3 AGB evolution

- The core is too cold for C or O to burn (neutrinos are cooling it!). See 7Ma-2425 in the "kippenhahn" panel.
- Furthermore, for stars lower than about 10 $M_{\odot}$, the carbon-oxygen core becomes degenerate (our models do not get to that point).
- Thus, the contracting core does not heat up the gas and the high internal temperatures needed to ignite the core are not reached.
- As the star reaches point continues, the shell-burning law has the core contracting (increasing in mass), the inner shell expanding, and the outer envelope contracting and increasing in effective temperature.
- But now the region between the two shell-burning sources has expanded sufficiently so that the temperature in the outer H-burning shell drops and extinguishes.
- Now there are only 2 distinct regions: the contracting core, and the now expanding envelope with He burning in between (shell-burning law).
- Figure 24.2 (bottom panel) shows this.

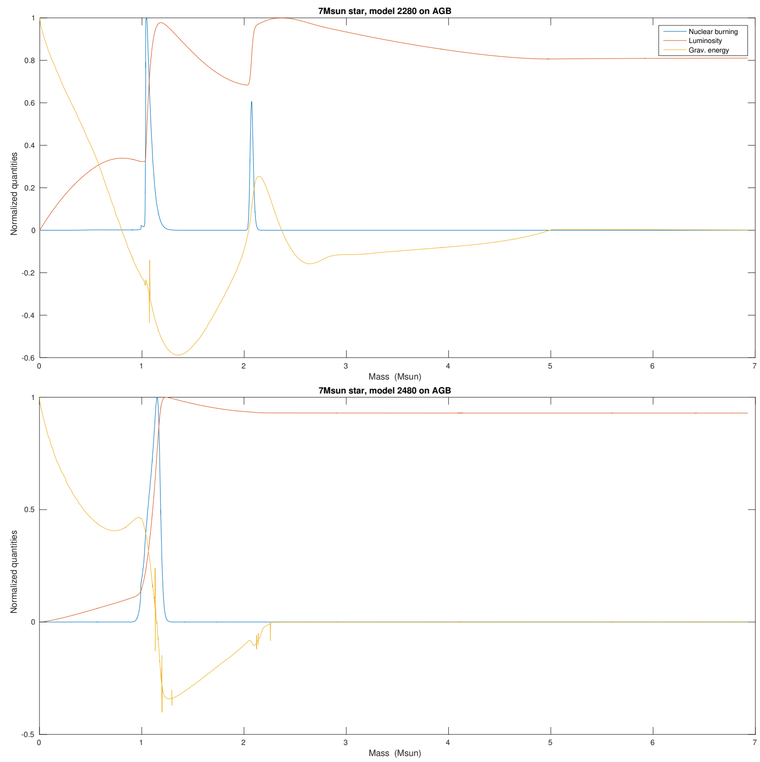

Figure 24.2: Interior profiles of the 7 $M_{\odot}$ star on the AGB during double-shell burning. The bottom panel is at a later stage when the outer-shell burning is extinguished.

- The luminosity increases as the CO core mass increases and contracts, and the star keeps climbing the AGB.
- A convective surface region develops (cooler surface temperature) and extends deeply, and "dredges up" processed material.
- It reaches down to the regions where the H shell had been burning for a while.
- Low-mass stars don't typically have a second dredge up since their H shell burning continues strongly.
- This dredge up brings a lot of material to the surface and reduces the mass size of the H-exhausted region.
- This is one reason why very massive white dwarfs are not formed.

24.4 Thermal pulses

- The following applies mostly for stars below 5 $M_{\odot}$.
- One sees that the growing He-burning shell approaches the bottom of the H-rich envelope.
- The He burning dies down a bit when it hits this region, contracts rapidly, and a H-shell reignites.
    1. As H burns, the He ashes fall onto the former shell burning region, and are compressed and heated.
    2. When the mass reaches about $10^{-3} M_{\odot}$ for a CO core mass of about $0.8 M_{\odot}$, He ignites again.
    3. A runaway occurs, in that this ignition heats the overlying shell burning region and causes it to burn even more violently (note the temp. dependence of nuclear reactions).

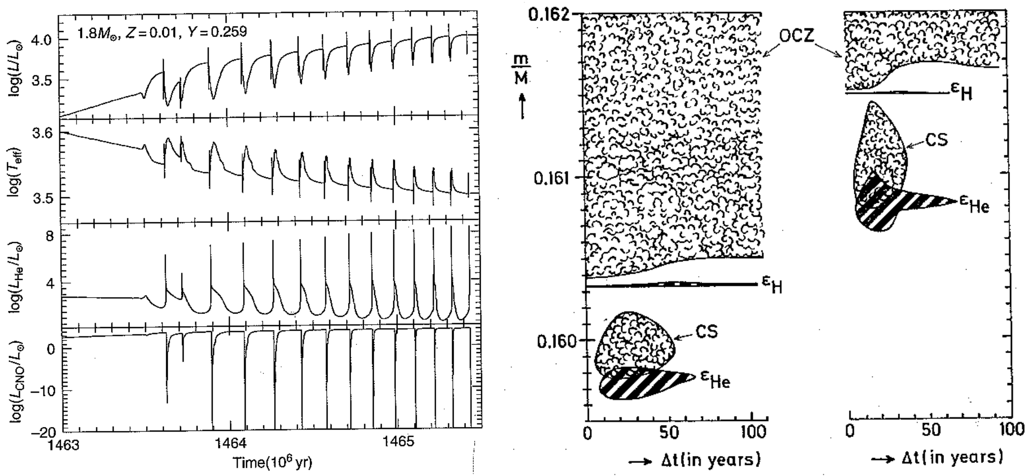

Figure 24.3: Thermal pulses. The left is for a $1.8M_\odot$ model showing many pulses with time. The right is for a $5M_\odot$ model

4. The luminosity of the He burning reaches very high values, and this causes the layers above it to expand.
5. The H-burning shell turns off.
6. Because of the high luminosity, a convection zone develops above the He-burning shell.
7. Eventually the convection helps expand the region, and the He burning drops strongly and cools.
8. The convection zone disappears as the luminosity decreases.
9. He burning continues, using up the He that the H-burning region produced before the flash.
10. As this source reaches the new discontinuity, a new H-burning region is created as before.
11. The He ash falls onto the He layer and the whole process happens again, now at a higher position.

- The timescale between pulses can be approximated roughly by

$$\log\tau \approx 3 + 4.5(1 - M_c/M_\odot). \tag{24.1}$$

- For a core mass of $0.5M_\odot$, this gives about 10^5 years; but drops to about 10 years for near-critical mass stars.
- Some stars can go through hundreds of pulses before the H shell gets depleted.
- In higher-mass stars, these cannot be observed because they are buried within the massive envelope.
- In low-mass AGB stars, the effects of the pulses can be seen chemically.
  - In the pulses when H burning is turned off, the surface convection zone moves inward, and a third dredge up can occur.
  - This can bring up carbon and heavy s elements (Sr, Y, Zr, Ba, La, Ce, Pr, Nd). Carbon-rich stars can be "produced."
  - Stars below about $1.5M_\odot$ will likely not go through a third dredge up, as their envelope mass isn't large enough.
  - For more massive stars, large amounts of lithium can be produced (from berrylium 7) and brought to the surface.

24.5 Production of s elements

- As mentioned earlier, AGB stars are spectroscopically enriched in s-process elements.
- The s refers to *slow* neutron captures (compared to β decay).
- About half of the elements heavier than Fe are created by this process.
- You need neutrons to make these heavier elements, where the neutron eventually decays into a proton to make a stabler isotope.
- The neutrons densities needed are lowish, of order $10v$ per unit volume.
- Since at this stage there is a large abundance of N-14 in the intershell region, it gets converted to Ne by
$$^{14}\mathrm{N}(\alpha,\gamma)^{18}\mathrm{F}(\beta^{+},\nu)^{18}\mathrm{O}(\alpha,\gamma)^{22}\mathrm{Ne}. \tag{24.2}$$
- If the temperatures at the base of the intershell region get to a few hundreds of MK, then $^{22}\mathrm{Ne}(\alpha,\mathrm{n})^{25}\mathrm{Mg}$ can provide a source of neutrons.
- For stars of initial mass $< 3M_{\odot}$, it probably does not get hot enough for this reaction to occur.
- There may be some channels through carbon 13 that provide a high neutron density for low-mass AGB stars.
- However, this is still a very active area of research due to many physical uncertainties.

Unit 25

Last Stages of Evolution: Low-Mass Stars

Here we discuss the late stages of evolution after the thermal pulses to the white dwarf.

25.1 Planetary Nebula

- AGB stars of low mass continute to brighten as their (outer) H shell approaches the surface.
- The thermal pulse number is set by the mass of the H envelope and the mass of the CO core.
- If the CO core exceeds about $1.4M_{\odot}$, non-degenerate, non-explosive carbon burning sets in and the AGB phase is over.
- Due to mass-loss processes during the AGB pulses, no star is really able to reach that core mass, because the H-burning shell stops when it's at about $10^{-3}M_{\odot}$ below the surface.
- The mass loss can be quite large, even $10^{-5}\,\mathrm{M_{\odot}\,yr^{-1}}$
- These superwinds that are created speed off at $10\,\mathrm{km\,s^{-1}}$
- The mass loss could be explained by pulsations (Mira variables).
- The gas compression, and subsequent cooling and formation of molecules and dust grains, can trap the outgoing radiation and get carried away.
- Anyway, however it happens, when the pulses stop, and the star evolves to hotter effective temperatures.
- The maximal luminosity depends on the star's initial mass (and that of its envelope) and how much mass it has lost.
- The H shell burning region approaches the surface and the effective temperature increases.
- An even faster stellar wind is produced, up to $2000\,\mathrm{km\,s^{-1}}$, which bumps up against the previous shell ejecta - this produces interesting features in the planetary nebula.
- The shell is dusty and optically thick (masers).
- The envelope is irradiated by UV radiation from remaining hot central star (core).

- The gas gets ionized and recombines quickly, giving distinct emission lines when the stellar remnant reaches about $30,000$K.
- A thin H-burning shell continues until the bluest point on the evolutionary track.
- Then, the H-rich envelope and He-rich layer contract quickly. A few scenarios are now possible:
  1. All nuclear burning shuts off and the star cools as a WD.
  2. The heating of the He from contraction leads to a thermal runaway, and the star goes back near to the AGB (born again). Then does the same stuff and cools as a WD.
  3. The heating of the envelope causes a H-burning runaway and the star is a *self-induced nova*. The process can be dynamic and blow off all H layers to become a DB white dwarf. Or, the process can be quiescent and it will burn H and start to cool down, possibly leading to another nova event.

25.2 White Dwarfs

- Recall all the discussion in Sec. 10.4.
- Degenerate matter obeys polytropic relations $P \sim \rho^\gamma$.
- For non-relativistic particles, $\gamma = 5/3$.
- For relativistic particles, $\gamma = 4/3$.
- The cores of evolved degenerate stars like white dwarfs are dominated by electron pressure rather than ion pressure.
- That's because $\mu_e \approx 2$ and $\mu_{\rm ion} \approx 12$, and $P \propto \mu^{-1}$.
- Why are white dwarfs special?
- Let's simply consider approximations to equilibrium with averages over the star, so

$$\frac{P}{M} = \frac{GM}{4\pi R^4}. \tag{25.1}$$

- For a polytrope, replacing density by its average value

$$P \sim \left(\frac{M}{R^3}\right)^\gamma. \tag{25.2}$$

- The "pressure" term f_p from equilibrium and the EOS, and the "gravity" term f_g from equilibrium, are

$$f_p \sim \frac{M^{\gamma-1}}{R^{3\gamma}}; \qquad f_g \sim \frac{M}{R^4}. \tag{25.3}$$

- Their ratio must be 1 for equilibrium

$$f = \frac{f_g}{f_p} \sim M^{2-\gamma} R^{3\gamma-4}. \tag{25.4}$$

- This is $M^{1/3}R$ for $\gamma = 5/3$, and $M^{2/3}$ for $\gamma = 4/3$.
- So consider a star less than some critical mass $M < M_{\rm crit}$ and non-relativistic electrons. The star can get into an equilibrium by just adjusting R so that $f = 1$.

- If we increase M so that $f > 1$ (more gravity), R must decrease to regain equilibrium (hence more massive WDs are smaller)
- Now consider relativistic electrons.
- We can only get equilibrium by setting the mass to a certain value $M = M_{\rm crit}$.
- If $M < M_{\rm crit}$, $f < 1$, and the pressure term is dominant and so the star expands so that the electrons become non relativistic.
- But if $M > M_{\rm crit}$ and $f > 1$, the gravity term forces the star to contract. But this does not help, because f is independent of R!
- The star collapses without "finding" an equilibrium.
- Clearly, $M_{\rm crit}$ is some limit.
- So again, consider a total degenerate equation of state then (recall Eq. 7.10)

$$P \approx \frac{R}{\mu_e}\rho T + K_\gamma \left(\frac{\rho}{\mu_e}\right)^\gamma . \tag{25.5}$$

- γ depends on density and relativistic effects, being $\gamma = 5/3$ for $\rho \ll 10^6$ and $\gamma = 4/3$ for $\rho \gg 10^6$.
- Using polytropic relationships once can derive a critical mass that governs the future behavior of the core of these dense stars

$$M_{\rm crit} = \left(\frac{K_{4/3}}{fG}\right)^{3/2} \mu_e^{-2}, \tag{25.6}$$

  where f is the ratio of the mean density to the central density.
- The cricital mass is then identified as the Chandrasekhar mass (Equation (10.43)):

$$\frac{M_{\rm Ch}}{M_\odot} = \frac{5.836}{\mu_e^2} = 1.456 \left(\frac{2}{\mu_{\rm e}}\right)^2 . \tag{25.7}$$

- It's also important to see how the central temperature and density depend on this critical mass:

$$\frac{\rho_c}{\mu_e} = \frac{1}{8}\left(\frac{K_{4/3}}{K_{5/3}}\right)^3 \left(\frac{M_c}{M_{\rm crit}}\right)^2 \approx 2.4 \times 10^5 \,{\rm g\,cm^{-3}} \left(\frac{{\rm M_c}}{{\rm M_{crit}}}\right)^2 , \tag{25.8}$$

$$T_c = \frac{1}{R}\frac{K_{4/3}^2}{K_{5/3}} \left(\frac{M_c}{M_{\rm crit}}\right)^{4/3} \approx 0.5 \times 10^9 \left(\frac{M_c}{M_{\rm crit}}\right)^{4/3} {\rm K}. \tag{25.9}$$

- For core masses below critical, maximum temperatures cannot exceed about 500 million K.
- In white dwarfs it is believed that the electrons are relativistic in the central part, but non-relativistic in the outer part.
- This changes the above results quantitatively, but not qualitatively.
- The mass of the core compared to the critical (Chandrasekhar) mass can be distinguised by 4 cases:

Case 1: If $M_c < M_{\rm crit} \approx M_{\rm Ch}$ and if there is no significant envelope (from mass loss or just a small original mass), so that M_c will NOT approach $M_{\rm Ch}$ during shell burning, then the core becomes degenerate, will cool, and the star becomes a white dwarf. T_c peaks. If it is a member of a binary system, then it can accrete enough mass to ignite carbon, which will detonate He and destroy the star in a runaway, producing at Type I supernova

Case 2: Initially if $M_c < M_{\rm crit}$ but there remains an envelope such that shell burning M_c can grow to $M_{\rm Ch}$, the core becomes degenerate and cools. However, ρ_c increases with M_c and carbon will be ignited. This will happen for $4 \lesssim M/M_\odot < 8$, and the stars will likely become white dwarfs as well, but C-O white dwarfs.

Case 3: If $M_{\rm crit} < M \lesssim 40M_\odot$, degeneracy does not happen. The core can heat up even more (because of no degeneracy) and further nuclear reactions can occur. Eventually the core collapses leading to formation of a neutron star and ejection of the envelope. This is a Type II supernova. See next section.

Case 4: If $M_c \geq 40M_\odot$, the core also will burn C non degenerately. Black hole. See next section.

- In reality, almost all stars born with about $8M_\odot$ or less will lose significant mass and will not reach the interior conditions to ignite carbon.
- Most WDs are observed at $0.6M\odot$ with very little variation.
- The higher-mass stars will lose all of their envelope and become CO WDs.

25.3 Futher WD properties

- It can be shown through simple Virial arguments that white dwarfs cool (approximately) according to the Mestel law

$$\Delta t \propto \left(\frac{L}{M}\right)^{-5/7} \approx \frac{4.5\times 10^7}{\mu_i}\left(\frac{LM_\odot}{L_\odot M}\right)^{-5/7} \quad \text{[year]}. \tag{25.10}$$

- The Δt represents the time for some change in luminosity.
- So higher mass WDs cool more slowly, due to more storage of energy.
- Increasing the ionic mean molecular weight decreases the evolutionary time, since there are fewer ions in that case.
- Roughly, for a WD to reach 1/1000th of the solar luminosity, it would take 1 billion years.
- A more precise cooling law would treat the ions more properly, since the steady decreae in temperature causes Coulomb interactions to become more important.
- The specific heats ratio grows and the ions form a lattice: this is *crystallization.*
- The crystallization obviously affects the EOS, and therefore the cooling times.
- The envelopes of WDs canonically predicta He layer of $\sim 10^{-2}M_{\rm tot}$ above the CO core, surrounded by a H envelope of mass $\sim 10^{-4}M_{\rm tot}$.
- Mass loss causes these notions to change.
- Metals are rarely observed due to atomic diffusion processes.
- Typically there are about 4 H WDs for every 1 non-H WD, but this varies widely with effective temperature, as evolutionary processes are still ongoing.
- The outer layers determine the opacity, and hence the cooling times.
- As WDs cool, they can develop convection zones as well.
- WDs can also be He core stars from an RGB progenitor that lost its envelope.
- There are also O-Ne WDs from higher-mass progenitors.

25.4 Type Ia supernovae

- Type I do not have visible hydrogen in its spectrum.
- The progenitor of these events are CO WDs that have accreted mass from a companion to exceed the critical mass .
- The companion is likely on the RGB or AGB where mass can be transferred effectively.
- In one scenario two similar stars evolve through a common envelope, and the two CO WDs merge through angular momentum loss from gravitational waves.
- The resultant object is higher than the critical mass.
- This is the *double-degenerate* channel.
- No observational evidence of these massive systems has yet to be shown.
- In the *single-degenerate* case, the events that happen depend strongly on the mass-accretion rates onto the low-mass star.
- For low rates, H burning on top of He layers produces an electron degenerate scenario, which can undergo explosive flashes.
- These are typically the classical *Novae*.
- Building up enough mass this way to reach the critical limit takes longer than a Hubble time, however.
- At moderate rates it can be shown that an explosion would occur for *sub-Chandrasekhar* mass objects.
- If enough He is accreted (either through H-burning above or from an He-rich companion), violent He-ignition can occur which can detonate the CO core.
- Basically, WDs are unable to regulate their temperature the way they did in earlier stages (expanding and cooling). So runaway fusion occurs.
- The explosion models must give about 10^{51}erg, as well as the production of heavy elements, such as lots of nickel.
- The explosive event is a shock wave whose speed depends on densities and abrupt thermal changes.
- Typical SNIa light curves rise rapidly and after about 20 days the light fades monotonically, with slighly different behavior in different bands.
- Almost all of them have a maximum absolute magnitude of -20 which declines linearly within 15 days after maximum.
- Thus, they are excellent standard candles.

Unit 26

Last Stages of Evolution: High-Mass Stars

26.1 Nuclear burning

- Now we consider stars greater than $15M_\odot$.
- If we only focus on the core, the processes in the late stages follow the simple causes and effects of

$$\ldots \rightarrow \text{nuclear burning} \rightarrow \text{fuel exhaustion} \rightarrow \text{core contraction} \rightarrow \text{core heating} \rightarrow \ldots \tag{26.1}$$

- At 500 million K, carbon burning can take place in the core, converting into Mg, Ne, and Na.
- Neutrinos from pair annihilation contribute a substantial amount to the luminosity (energy loss), even more than the nuclear burning.
- So the star contracts to make up the loss with gravitational energy.
- The convective C core is less massive for more massive stars, as neutrinos cause a considerable amount of mass loss.
- Eventually the burning moves to a shell of C and s-elements are produced.
- The core is now about 70% O, 25% Ne, and the rest Mg.
- Ne burning sets in .
- Oxygen burning can set in at 1.5 billion K.
- Eventually, the photons released in such reactions have such high energies that they can photo-disassociate surrounding nuclei.
- A large number of neutrons begin to be produced.
- Silicon burning sets in through interactions with alpha particles.
- Then it moves into the shell.
- The process stops at the iron group, and the last reaction is ^{52}Fe capturing an alpha particle to make nickel.
- Further reactions would **require** energy to proceed. All reactions now are balanced by their inverse reaction.

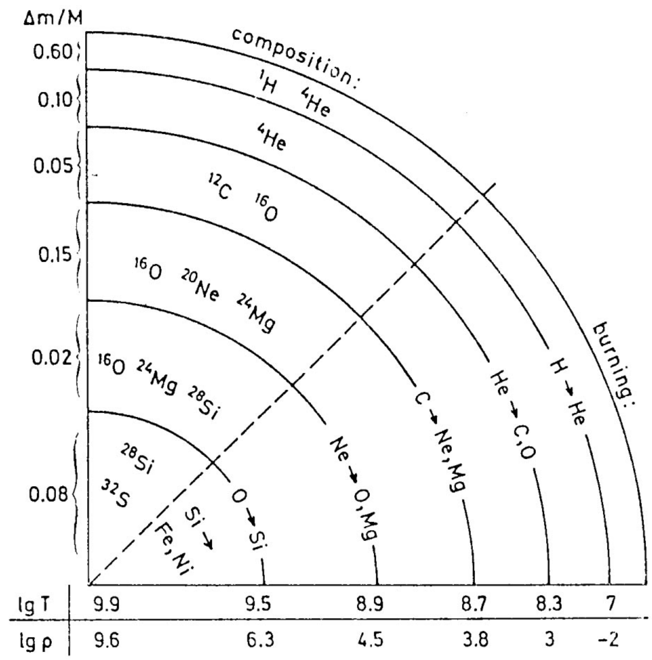

Figure 26.1: Illustration of the "onion-skin" structure in the interior of a highly evolved massive star. From Kippenhahn and Weigert [1990].

- Figure 26.1 shows how the core builds up its layers this way.
- For a $15M_{\odot}$ star, the burning of each successive element is rapid:

$$\mathrm{H}(10^7); \mathrm{He}(10^6); \mathrm{C}(10^3); \mathrm{Ne}(10^1); \mathrm{O}(10^1); \mathrm{Si}(10^{-1}), \tag{26.2}$$

  where the times are in years.
- Since all this happens so quickly, the surface is basically "frozen in" and the star does not move from right to left until it explodes.

26.2 Type II supernova - core collapse

- The core of the massive star is hot, $T_9 \approx 10$, and electrons are relativistic.
- Simply, the ratio of specific heats, γ, drops below 4/3 and the star is in an unstable configuration.
- After silicon burning, electrons are captured by protons

$$p^+ + e^- \longrightarrow n + \nu_e, \tag{26.3}$$

  producing a lot of electron neutrinos and neutrons.
- Neutrinos carry away energy, cool the core, and pressure drops.
- Photo-disintegtration produces many free α particles.

$$^{56}\mathrm{Fe} \longrightarrow 13\,^4\mathrm{He} + 4\mathrm{n} - 100\ \mathrm{MeV}. \tag{26.4}$$

- The loss of free electrons also reduces the pressure.

- The core collapses from overlying weight on the scale of a few seconds.
- This collapse halts when the neutron degeneracy pressure kicks in ($\rho \approx 10^{15}\,\mathrm{g\,cm^{-3}}$) - note that this is nuclear matter density! And 40km in size!
- Energy release of about 10^{53}erg from change in gravitational energy $GM^2/\Delta R$. Where does it go?
- This is as much light as a galaxy shines at for decades.
- In one scenario, most of the light is not released however, but goes into the kinetic energy of a shock.
- This propagates outward into the outer core region that is still collapsing.
- Naively, this might blow off the outer layers, but that does not happen.
- In the more accepted scenario, the increased core density causes it to be optically thick to neutrinos.
- They begin to deposit their energy into the material.
- This causes the outward shock that blows off the star's layers.
- This is the core-collapse Type II event.
- Heavy nuclei are created through neutron capture (s and r processes)
- Just to note, the neutrino cross section is extremely small, and its mean free path is

$$\ell_\nu = \frac{1}{n\sigma_\nu} \approx \frac{1}{\mu_e A}\left(\frac{\rho}{\mu_e}\right)^{-5/3} 1.7\times 10^{25}\,\mathrm{cm}. \tag{26.5}$$

- $\mu_e = 2$, $A = 100$, and a density of about 10^{10}, $\ell_\nu \approx 10^7$cm, which is contained withing the collapsing core
- So neutrinos do not escape without interaction.
- About 1% of the energy goes into the outward motion, and 1% of that gets released as photons
- So only about 10^{49} erg of energy gets radiated over a few months
- We observe these supernova because hydrogen lines are present.
- The collapse occured when a H-rich envelope still existed.
- If the initial star was $M \geq 25M_\odot$, the remnant is likely a neutron star.
- For higher masses, it is too much to be supported by neutron degeneracy pressure.
- The object then becomes a **black hole**.
- If the initial star was over $100M_\odot$, or a core He mass of about $40M_\odot$, a different scenario might take place.
- After He burning the thermal environment produces electron-positron pairs, reducing the specific heat so that $\gamma < 4/3$.
- The star immediately starts to collapse and subsequent burning is not enough to halt the collapse.
- The star produces a black hole in a *pair-instability* supernova.

26.3 Neutron star

- Masses between 1.2 and $2.5M_\odot$ and $R \approx 10$km
- The mass-radius relationship for neutron stars in each case is (derived from our polytropic equations):

$$M = \left(\frac{15.12\,\mathrm{km}}{\mathrm{R}}\right)^3 \mathrm{M}_\odot;\ \mathrm{non-relativisitic} \tag{26.6}$$

$$M = 5.73\mathrm{M}_\odot \equiv \mathrm{M}_{\mathrm{Ch}}^{\mathrm{NS}};\ \mathrm{relativisitic} \tag{26.7}$$

- However, tne maximum mass of a neutron star depends on the existence of a general-relativistic instability (interactions between nucleons)
- This likely takes place well before the M_{Ch} for a neutron star, hence the $\sim 3M_\odot$ limit
- Most neutron star observations are in a very narrow range of masses $M_{\mathrm{ns}} = 1.35 \pm 0.04M_\odot$.
- The neutron-degenerate material creates mostly an isothermal environment.
- They cool down faster than WDs.

26.4 Black hole

- For Type II remnant $> 2.5M_\odot$, or a progenitor $> 25M_\odot$, a black hole is produced.
- The radius, determined from the escape velocity, is:

$$R = 2\frac{GM}{c^2} = 2.95 \times 10^5 \frac{M}{M_\odot}\ [\mathrm{cm}]. \tag{26.8}$$

Unit 27

Instability Strip and Pulsations

27.1 Background

- We first consider the types of stars that display pulsations.
- Figure 27.1 shows where many of these types of stars lie on the H-R diagram.
- The Cepheid instability strip is the main thing to pay attention to, which spans from high luminosity down to the main sequence.
- Here, there are Cepheids, RR Lyrae, and δ-Scuti stars, etc.
- The instability strip is very narrow, and we've learned that it's hard to "catch" stars in the Hertzsprung Gap.
- The higher luminosity stars must therefore be more massive, going through their "loops," which occur on a much longer timescale.
- These stars have pulsations excited by an opacity mechanism, explained later.
- Typically only one mode is observed for the more luminous stars (Cepheids and RR Lyrae), a radial mode.
- For the stars near the main sequence, such as the δ Scuti, γ Dor, slowly pulsating B stars (SPB) and β Cephei, multiple modes are observed.
- δ Scutis have masses around $1.5 - 2.5M_{\odot}$ (convective cores, F stars), and have periods of an hour or so.
- This makes their observation a bit difficult.
- They are expected to have acoustic pulsations.
- The Mira variables are related to the AGB stage of stars.
- The "EC14026" stars are subdwarf B variables (SdB), believed to also pulsate from an opacity mechanism (related to iron).
- These interesting stars are on the horizontal branch as core helium burning is proceeding.
- However, at 35,000K they are much bluer than most HB stars.
- They have somehow lost most of their H envelope beforehand.

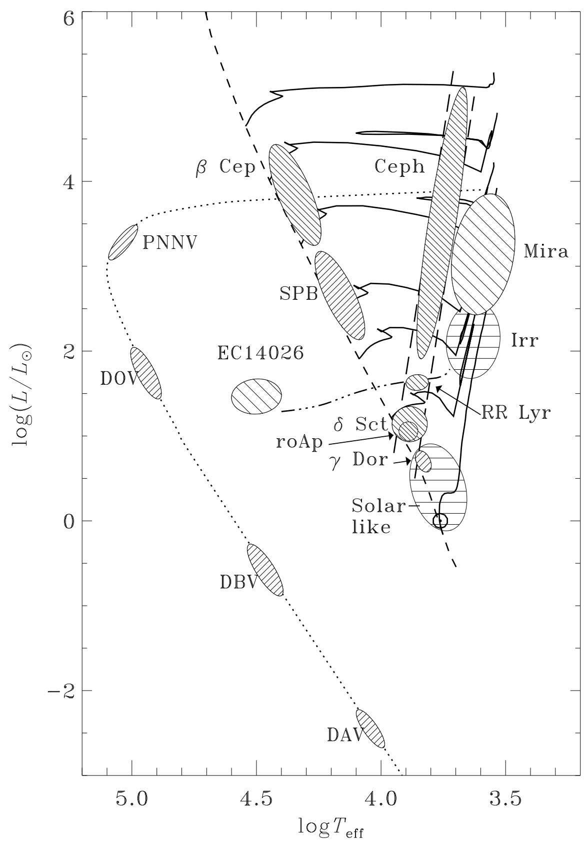

Figure 27.1: Schematic Hertzsprung-Russell diagram illustrating the location of several classes of pulsating stars. The dashed line shows the zero-age main sequence, the continuous curves are selected evolution tracks, at masses 1, 2, 3, 4, 7, 12 and $20M_\odot$, the dot-dashed line is the horizontal branch and the dotted curve is the white-dwarf cooling curve. From Christensen-Dalsgaard [2003].

- The "Irr" irregular pulsators show strange amplitude variations, but are now known as red giant solar-like pulsators.
- White dwarfs pulsate all along the cooling phase (ZZ Ceti as they are sometimes known).
- They are pulsating with periods up to 10 min, much longer than their dynamical timescales.
- Likely pulsating with internal gravity modes.
- Finally, solar-like oscillations are expected in stars < 7000K, due to convective mechanisms operating near the surface.
- Pulsations can be observed by intensity fluctuations, doppler-velocity observations, and line-profile variations (for nonradial modes).

27.2 Pulsation mechanisms

- The adiabatic sound speed of stellar interiors is given as

$$c_s^2 = \frac{\gamma p}{\rho}, \tag{27.1}$$

where again

$$\gamma = \frac{\mathrm{d}\ln p}{\mathrm{d}\ln \rho}.$$

- The most fundamental period of pulsations is inversely proportional to the mean density of a star

$$\Pi \propto \overline{\rho}^{-1/2}. \tag{27.2}$$

- This can be obtained from hydrostatic equilibrium and some basic principles, but can more easily be seen just from the dynamical timescale, Eq. (1.11):

$$t_{\mathrm{dyn}} = \left(\frac{R^3}{GM}\right)^{1/2} \propto \overline{\rho}^{-1/2},$$

  which is its direct counterpart.
- A rigorous treatment can provide an exact equation for Eq. (27.2).
- Radial modes are standing waves, and the fundamental mode has only a node at the center and the surface of the star
- After that, we have overtones of the fundamental mode, with nodes placed internally to the star
- Cepheids are observed to pulsate in fundamental radial modes
- Their magnitude, temperature, radius, and surface velocity change as a function of time
- The temperature varies by about 1000 K
- The radius by about 10%
- The star is brightest when it is *expanding* after its minimum radius was reached
- There is a phase lag between the radius and the luminosity
- We do know that as stars pulsate, energy is lost in each pulsation cycle, as most of the volume of the star damps it
- These pulsations can only continue of there is a driving mechanism that is feeding energy into the pulsation, at least as much that has been damped out
- The first proposition to explain such pulsations was that of a heat engine (convert thermal to mechanical energy)
- Pulsations are driven if positive work is done, and damped if negative work is done (on a layer, and throughout the star)
- Eddington postulated a valve mechanism, whereby a layer of a star could block energy that would then push the layer outward.
- After expansion, the blocked region would allow energy to pass through and the cycle would start again
- So, simplistically, the driving mechanism is this
  - The increase in opacity due to ionization halts the energy flow
  - The absorbed energy ionizes and eventually heats the gas and the pressure increases, pushing the local layer outward past its equilibrium point (expansion)
  - This ionization process reduces the opacity since now the density decreases (expansion) and the temperature does not decrease too rapidly because of partial recombination ($\kappa \propto \rho T^{-3.5}$)
  - Radiation can now flow freely, the gas cools and can no longer support the overlying weight

- The star contracts and compresses and the density increases
- The opacity is thus raised again and the whole process stars over

- This is the κ mechanism
- The other main driving mechanism is stochastic driving due to convection near the surface
- These modes are typically stable, but there is a lot of acoustic energy in outer convection zones such that this noise is transferred to the energy of global oscillations

27.3 Ionization zones

- The ionization zones being considered are hydrogen and the first and second ionization levels of helium
- These occur at about 10,000-20,000K
- To fully ionize helium requires about 40,000K
- For hot stars, with a $T_{\text{eff}} \geq 7500$K, these ionization zones are near the surface
- There is not much mass here and oscillations cannot be driven strongly
- In cooler stars with $T_{\text{eff}} \lesssim 5500$K, these zones occur deeper
- However, convection is occuring in these envelopes
- Convection is efficient and does not allow a sufficient "blocking" of the radiation or energy flux
- Pulsations may not therefore occur from this mechanism, and the red edge of the instability strip is defined
- Most of the driving is due to ionizing He zones from modeling
- The H ionization zones are interesting and since they lie above the He zones, they "carry" the emergent luminosity outward
- Thus the phase lag in the luminosity-minimum radius relations
- Note that these ionization zones alter the value of Γ_1, and lower the adiabatic gradient.
- These regions can often be unstable to convection which can also drive pulsations through convective flux
- For β Cephei stars, it is Fe-group elements doing the driving

Appendix A

Conduction

A.1 Eddington Luminosity

- Let's take a quick stop and look at an interesting consequence of diffusive radiation.
- Consider the near-surface of a star where radiation into space dominates over the gas pressure
- We found before that

$$P_{\rm rad} = \frac{1}{3} a T^4,$$

so

$$\frac{{\rm d}P_{\rm rad}}{{\rm d}T} = \frac{4}{3} a T^3 \frac{{\rm d}T}{{\rm d}r}. \tag{A.1}$$

- The radiative flux can then be expressed as (see Equation (13.8))

$$F_{\rm rad} = -\frac{c}{\kappa_{\rm R}\rho} \frac{{\rm d}P_{\rm rad}}{{\rm d}r}. \tag{A.2}$$

- Let's assume that it might be possible that the radiation pressure overcomes gravity.
- In that case consider hydrodstatic (non)equilibrium to occur when

$$-\frac{{\rm d}P_{\rm rad}}{{\rm d}r} > \rho g. \tag{A.3}$$

- Using Equation (A.2) this becomes

$$\frac{F_{\rm rad}\kappa_{\rm R}}{c} > g. \tag{A.4}$$

- If we just consider the area near the surface and look and consider $L = 4\pi R^2 F_{\rm rad}$, then

$$\frac{\kappa_{\rm R} L}{4\pi c G M} > 1. \tag{A.5}$$

- We can write the quantity on the left as

$$\frac{\kappa_{\rm R} L}{4\pi c G M} = 7.8 \times 10^{-5} \kappa_{\rm R} \left(\frac{L}{L_\odot}\right) \left(\frac{M}{M_\odot}\right)^{-1}, \tag{A.6}$$

which shows that this number is typically small, much less than 1.

- For massive stars, however, the luminosities can get quite large.

- So if we define the Eddington luminosity at the point where this number is unity, then

$$\frac{L_{\rm Edd}}{L_\odot} \approx 3.7 \times 10^4 \left(\frac{M}{M_\odot}\right), \tag{A.7}$$

where we used $\kappa = 0.34$.

- If the Eddington luminosity starts to approach a good fraction of this value, then equilibrium is lost and severe mass loss occurs.

A.2 Conduction

- Degenerate electrons (in white dwarfs or supergiant cores) are the primary carrier for energy transport in such objects.
- The process is again diffusion of the Fick's Law type

$$F_{\rm cond} = -D_{\rm e}\frac{{\rm d}T}{{\rm d}r}. \tag{A.8}$$

- The diffusion coefficent $D_{\rm e}$ can again be expressed by an "opacity" of sorts, $\kappa_{\rm cond}$, and put into a form similar to Equation (13.8)

$$F_{\rm cond} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\rm cond}\rho}\frac{{\rm d}T}{{\rm d}r}. \tag{A.9}$$

- Assume we can compute $\kappa_{\rm cond}$. Then the total energy flux in the star (so far) would be

$$F_{\rm tot} = F_{\rm rad} + F_{\rm cond} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\rm tot}\rho}\frac{{\rm d}T}{{\rm d}r}, \tag{A.10}$$

where

$$\frac{1}{\kappa_{\rm tot}} = \frac{1}{\kappa_{\rm R}} + \frac{1}{\kappa_{\rm cond}}. \tag{A.11}$$

- Note again how opacities are not simply "additive," as the fluxes are.
- Realize that whatever opacity is smaller is the one that contributes most to the total opacity and thus determines the energy flux (or lack thereof).
- For example, in typical non-degenerate stellar matter, $\kappa_{\rm cond}$ is large (so conduction is negligible), and radiative opacities dominate. Think of it as the one with biggest "channel" that lets the heat through.
- From solid-state physics, one can show that

$$\kappa_{\rm cond} \approx 4 \times 10^{-8}\frac{\mu_e^2}{\mu_{\rm I}}Z_c^2\left(\frac{T}{\rho}\right)^2. \tag{A.12}$$

- In a white dwarf, one may encounter $\rho \approx 10^6\,{\rm g\,cm^{-3}}$ and $T \approx 10^7{\rm K}$, made of carbon.
- The radiative opacity in this environment $\kappa_{\rm R} \approx 0.2\,{\rm cm^2\,g^{-1}}$ (Equation (14.4)). With $\mu_e = 2$, $\mu_{\rm I} = 12$, and $Z_c = 6$, we find $\kappa_{\rm cond} \approx 5 \times 10^{-5}\,{\rm cm^2\,g^{-1}}$.
- Thus the total opacity is dominated by radiative processes, and so the flux is carried out by conduction.

Appendix B

The Virial Theorem

- The Sun is in hydrostatic equilibrium. It is not necessarily in thermal equilibrium, but let's see.
- First a quick look at numbers. Consider the **thermal** energy (internal energy density) of an ideal monatomic gas

$$u = \frac{3}{2}\frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}}. \tag{B.1}$$

- Integrated over the star $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ and looking at the specific internal energy $\mathrm{d}U = u\mathrm{d}V$, we find

$$U = \int_0^R \left(\frac{3}{2}\frac{k_{\mathrm{B}}}{\mu m_{\mathrm{u}}}T\right)\rho 4\pi r^2 \mathrm{d}r = \int_0^M \left(\frac{3}{2}\frac{k_{\mathrm{B}}}{\mu m_{\mathrm{u}}}T\right)\mathrm{d}m = \frac{3}{2}\frac{k_{\mathrm{B}}}{\overline{\mu} m_{\mathrm{u}}}\overline{T}M, \tag{B.2}$$

where the overbars denote average values (we ignore radial dependence of temp. and composition, for now). This is dirty.

- Now consider the gravitational potential energy

$$E_{\mathrm{G}} = -\int_0^M \left(\frac{Gm}{r}\right)\mathrm{d}m \approx -\frac{GM^2}{R}. \tag{B.3}$$

- Using some typical solar values values: $k_{\mathrm{B}} = 1.38 \times 10^{-16}\mathrm{erg\,K^{-1}}$, $\overline{\mu} = 0.62$, $m_{\mathrm{u}} = 1.66 \times 10^{-24}\mathrm{g}$, mean temperature $\overline{T} = 10^7\mathrm{K}$, $M_\odot = 1.99 \times 10^{33}\mathrm{g}$, $G = 6.67 \times 10^{-8}\mathrm{dyne\,cm^2\,g^{-2}}$, $R_\odot = 6.96 \times 10^{10}\mathrm{cm}$. We find that

$$U \approx +4.0 \times 10^{48}\,\mathrm{erg}, \tag{B.4}$$
$$E_G \approx -3.8 \times 10^{48}\,\mathrm{erg}. \tag{B.5}$$

- Why are these two numbers so close, even the same order of magnitude? This suggests something deeper is happening.
- To find out, let's take the expression for hydrostatic equilibrium, Equation (9.3), multiply by $4\pi r^3$ on both sides and integrate over the whole star:

$$\int_0^R 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = -\int_0^R \rho\frac{Gm}{r^2}4\pi r^3 \mathrm{d}r. \tag{B.6}$$

- Before we proceed, recall from Sec. 6.3 and later that a generalized expression relating pressure and internal energy is

$$P = (\gamma - 1)u = (\gamma - 1)\rho U, \tag{B.7}$$

which you can prove to yourself works for different types of gases. This will be used below.

- Integrate left side of Equation (B.6) by parts, and set $P(R_\odot) = 0$. This gives

$$\begin{aligned}
-\int_0^R 12\pi P r^2 \mathrm{d}r &= -\int_0^R \frac{Gm}{r^2} 4\pi\rho r^3 \mathrm{d}r \\
-\int_0^R 12\pi(\gamma-1)U\rho\, r^2 \mathrm{d}r &= -\int_0^R \frac{Gm}{r^2} 4\pi\rho r^3 \mathrm{d}r \\
-\int_0^R 3(\gamma-1)U\, 4\pi\rho r^2\, \mathrm{d}r &= -\int_0^R \frac{Gm}{r} 4\pi\rho r^2 \mathrm{d}r \\
-\int_0^R 3(\gamma-1)U\, \mathrm{d}m &= -\int_0^R \frac{Gm}{r}\, \mathrm{d}m \\
-\zeta U_\mathrm{T} &= E_\mathrm{G},
\end{aligned}$$

  or

$$E_\mathrm{G} + \zeta U_\mathrm{T} = 0, \tag{B.8}$$

  where we let $\zeta = 3(\gamma - 1)$, and assume that it is constant throughout a star (which is a good approximation in most of the interior). We are also calling U_T the total internal energy of the star.

- Equation (B.8) is a generalized Virial theorem, relating the gravitational and internal energies.

- For an ideal gas, $\gamma = 5/3$, $\zeta = 2$, and $E_\mathrm{G} = -2U_\mathrm{T}$.

- For a photon gas with $\gamma = 4/3$, $E_\mathrm{G} = -U_\mathrm{T}$.

- Taking a look at the total energy

$$E_\mathrm{T} = E_\mathrm{G} + U_\mathrm{T} = (1-\zeta)U_\mathrm{T} = \frac{\zeta - 1}{\zeta} E_\mathrm{G}. \tag{B.9}$$

- If we want a bound system $E_\mathrm{T} < 0$, we need $\zeta > 1$, or, $\gamma > 4/3$. We'll come back to this later.

- To maintain hydrostatic equilibrium, the star will slowly shrink and expand and these 2 energies will change, making the total energy not constant. Thus, the gas must radiate, so

$$\frac{\mathrm{d}E_\mathrm{T}}{\mathrm{d}t} + L = 0 \tag{B.10}$$

  is always true.

- For an ideal gas,

$$L = (\zeta - 1)\frac{\mathrm{d}U_\mathrm{T}}{\mathrm{d}t} = \frac{\mathrm{d}U_\mathrm{T}}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}E_\mathrm{G}}{\mathrm{d}t} \simeq -\frac{1}{2}\frac{GM}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t}. \tag{B.11}$$

- For a contracting star, U_T increases and so does the temperature of the gas.

- Therefore, upon contraction for example, half the energy is radiated away, and the other half is used to heat the star.

- Keep in mind that it is the luminosity that is causing the star to shrink. Anything with a finite temperature will radiate, and so $\dot{E}_\mathrm{T} \neq 0$.

- Also note that these systems *become warmer as they lose heat!*

- From Eq. (B.11), we see that

$$\frac{\mathrm{d}R}{\mathrm{d}t} \simeq -2\frac{R}{\tau_\mathrm{KH}}, \tag{B.12}$$

  where τ_KH is the Kelvin-Helmholtz, or thermal timescale from Eq. (1.15).

$m/m_\odot$	$r/r_\odot$	$T[K]$
0.000	0.000	1.5×10^7
0.125	0.124	1.2×10^7
0.250	0.170	1.0×10^7
0.375	0.210	8.9×10^6
0.500	0.254	7.7×10^6
0.625	0.306	6.6×10^6
0.750	0.367	5.4×10^6
0.875	0.470	4.2×10^6
1.000	1.000	5.8×10^3

Figure B.1: Tabulated solar model values. Use in Example Problems B.1 and B.2.

- So this timescale is the characteristic time over which a star contracts gravitationally, as well as the characteristic time for the radiation of a star's thermal energy. Changes that involve significant energy losses (or gains) cannot happen over a shorter timescale if hydrostatic equilibrium is maintained.
- If changes happen to occur more quickly, they must happen nearly adiabatically, involving small energy changes.

EXAMPLE PROBLEM B.1: Compute the actual gravitational energy of the Sun $E_{\rm G}$. To do this, we need to calculate the total work required to disperse the solar matter over distances $r \gg R_\odot$. Use the table of solar model values in Figure B.1. How does this energy compare to the total solar irradiance ($1.39 \times 10^6 \mathrm{erg\,s^{-1}\,cm^{-2}}$ at 1 AU) or to a solar flare ($\approx 10^{33}$ J)?

Answer: Consider a spherical shell of mass $\mathrm{d}m$ at radius r and the mass interior to r, m. The force acting on this shell from all interior mass is

$$F(r) = -G\frac{m\mathrm{d}m}{r^2}.$$

The work (energy) required to take this shell and carry it to infinity:

$$\begin{aligned} E_{\rm G}^{\rm 1\,shell} &= \int F(r')\,\mathrm{d}r' \\ &= -\int_r^\infty \frac{Gm\mathrm{d}m}{r'^2}\mathrm{d}r' \\ &= -\frac{Gm\mathrm{d}m}{r}. \end{aligned}$$

Now we need to consider removing all shells (all the mass of the Sun) and rewrite it in a form so the table of values can be used:

$$\begin{aligned} E_G = \int_{\rm all\,shells} E_{\rm G}^{\rm 1\,shell} &= -\int_0^{M_\odot} \frac{Gm}{r}\mathrm{d}m \\ &= -\frac{GM_\odot^2}{R_\odot}\int_0^{M_\odot} \frac{m}{M_\odot}\frac{R_\odot}{r}\mathrm{d}\left(\frac{m}{M_\odot}\right) \\ &= -\frac{GM_\odot^2}{R_\odot}\sum_i \frac{m_i}{M_\odot}\frac{R_\odot}{r_i}\Delta\left(\frac{m_i}{M_\odot}\right) \\ &= -\frac{GM_\odot^2}{R_\odot}\cdot 0.125\,[0 + 1.008 + \ldots] \\ &\simeq -1.6476\frac{GM_\odot^2}{R_\odot} \\ &\simeq -6.25 \times 10^{48}\mathrm{erg} \end{aligned}$$

Note that we did a similar problem earlier and just integrated without knowing any radial dependence of the quantities, and found $E_G \approx -3.8 \times 10^{48}$erg. In this way, we are more precise.

The solar irradiance is $1.39 \times 10^6\,\mathrm{erg\,s^{-1}cm^{-2}}$ at $1\,\mathrm{AU} = 1.496 \times 10^{13}\,\mathrm{cm}$. So the total irradiance at the solar surface, or luminosity (integrated over the $4\pi R^2$) is $3.87 \times 10^{33}\,\mathrm{erg\,s^{-1}}$. This is the energy per second in radiation. So, considering graviational energy only, the Sun can provide $6.25 \times 10^{48}/3.87 \times 10^{33} = 1.615 \times 10^{15}\,\mathrm{s} = 5.12 \times 10^7$ year of irradiance. Since the Sun is over 10^9 years old, it must have some other energy source.

EXAMPLE PROBLEM B.2: Suppose the Sun is an ideal monatomic gas in hydrostatic equilibrium. Calculate the internal energy and compare to the gravitational energy, i.e., rederive the Virial theorem. Use the Virial theorem to find the mean mass-weighted temperature and compare to the tabulated value. Hints: Start with hydrostatic equilibrium, but it's convenient to be in terms of $\mathrm{d}P$ and $\mathrm{d}m$. Multiply both sides by $V = 4/3\pi r^3$ and integrate both sides. You will eventually make use of

$$U = \int_0^{M_\odot} c_V T \mathrm{d}m, \tag{B.13}$$

(since $\mathrm{d}U = u\mathrm{d}V$), but you can consider the specific heat constant. Recall $c_V = 3/2(R/\mu)$. Ultimately you will end up with $E_G + 2U = 0$. Finally you want to find the mass-weighted temperature, given as

$$\langle T \rangle = \frac{1}{M_\odot} \int_0^{M_\odot} T \mathrm{d}m. \tag{B.14}$$

Then use Table B.1 as you have before to find the value.

Answer: Let's start with hydrostatic equilibrium, a balance between pressure changes and gravity, Equation (9.3):

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}.$$

We also use Equation (9.6) to get

$$\mathrm{d}P = -\frac{Gm}{4\pi r^4}\mathrm{d}m.$$

Now let's multiply both sides by the volume of a sphere:

$$\begin{aligned} \mathrm{LHS} &= V\mathrm{d}P = \mathrm{d}(VP) - P\mathrm{d}V \\ \mathrm{RHS} &= -\frac{Gm}{4\pi r^4}\mathrm{d}m\frac{4}{3}\pi r^3 = -\frac{Gm}{3r}\mathrm{d}m. \end{aligned}$$

Now integrate both sides from the center to the surface of the Sun. The first term on the LHS vanishes because $P = 0$ at the surface and $V = 0$ at the center. Use $P = \rho RT/\mu = 2/3\rho c_V T$. What's left:

$$\begin{aligned} \mathrm{LHS} &= -\int_c^s P\mathrm{d}V \\ &= -\int_c^s \frac{2}{3}\rho c_V T \mathrm{d}V \\ &= -\frac{2}{3}\int_0^{M_\odot} c_V T \mathrm{d}m \\ &= -\frac{2}{3}U \\ \mathrm{RHS} &= -\int_c^s \frac{Gm}{3r}\mathrm{d}m \\ &= \frac{1}{3}E_G. \end{aligned}$$

We thus have

$$E_G + 2U = 0.$$

The mass weighted temperature is

$$\langle T \rangle = \frac{1}{M_\odot}\int_0^{m_\odot} T\mathrm{d}m.$$

We just found that the virial theorem gives (for constant c_V)

$$2c_V \int_0^{M_\odot} T \mathrm{d}m = -E_G.$$

Inspecting the above 2 equations shows that

$$2c_V M_\odot \langle T \rangle = -E_G.$$

Then

$$\begin{aligned}
\langle T \rangle &= -\frac{E_G}{2c_V M_\odot} \\
&= -\frac{E_G \mu m_\mathrm{u}}{3k_\mathrm{B} M_\odot} \\
&= \frac{(6.25 \times 10^{48}\,\mathrm{erg})(0.62)(1.66 \times 10^{-24}\,\mathrm{g})}{(3)(1.38 \times 10^{-16}\,\mathrm{erg\,K^{-1}})(1.99 \times 10^{33}\,\mathrm{g})} \\
&= 7.81 \times 10^{6}\,\mathrm{K}
\end{aligned}$$

Using the tabulated values we get

$$\langle T \rangle_\mathrm{tab} = \int_0^{M_\odot} T \mathrm{d}\left(\frac{m}{M_\odot}\right) = \sum_i T_i \, \Delta\left(\frac{m_i}{M_\odot}\right) = 6.85 \times 10^{6}\,\mathrm{K}.$$

Therefore, we see the Sun is in a very close hydrostatic and Virial equilibrium.

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