# Unit 18

# Theory of the Main Sequence

## 18.1 Summary of stellar structure

- Mass and radius relationship
$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho$$
- Hydrostatic equilibrium
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$$
- Energy
$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \varepsilon$$
- Energy transport
$$L = 4\pi r^2 (F_{\mathrm{rad}} + F_{\mathrm{conv}})$$
  - Radiation
$$F_{\mathrm{rad}} = -\frac{4acT^3}{3\kappa_{\mathrm{R}}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$
  - Convection
$$F_{\mathrm{conv}} = \rho C_p T (g\delta)^{1/2} \frac{\ell^2}{4} H_p^{-3/2} (\nabla - \nabla_{\mathrm{ad}})^{3/2}.$$
- Equation of state
$$P = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}} + \frac{a}{3} T^4$$
- Rosseland opacity
$$\kappa_{\mathrm{R}} = \kappa_{\mathrm{R}}(\rho, T, \mu)$$
- Energy generation
$$\varepsilon = \varepsilon(\rho, T, \mu) = \varepsilon_{\mathrm{nuc}} + \varepsilon_{\mathrm{grav}}$$
where
$$\varepsilon_{\mathrm{nuc}} = \varepsilon_0 \rho T^n, \tag{18.1}$$
and
$$\varepsilon_{\mathrm{grav}} = -T\frac{\mathrm{d}S}{\mathrm{d}t}$$
is related to Kelvin-Helmholtz contraction (and note the time dependence).

- Abundance changes

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= -\frac{\varepsilon_{\rm nuc}}{26.7\,\mathrm{MeV}} \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= -\frac{\mathrm{d}X}{\mathrm{d}t}\end{aligned}$$

- Boundary conditions

$$\begin{aligned}r &\longrightarrow 0; \quad m(r) \longrightarrow 0 \\ r &\longrightarrow 0; \quad L(r) \longrightarrow 0 \\ r &\longrightarrow R; \quad m(r) \longrightarrow M \\ r &\longrightarrow R; \quad \rho(r) \longrightarrow 0 \\ r &\longrightarrow R; \quad T(r) \longrightarrow T_{\rm eff}\end{aligned}$$

- Vogt-Russell Theorem
  - There are only 2 free parameters in the equations needing to be solved: the total stellar mass $M$ and the chemical composition.
  - The theorem states that just these 2 parameters uniquely determine the structure of any star. Then, evolution is only based on the changing of the composition due to nuclear burning.

## 18.2 Homology relations for stars in radiative equilibrium

- Solving the equations of stellar structure is possible, yet difficult.
- There are ways of obtaining useful insights without doing so (like using the first 2 equations to study polytropes).
- This involves basic unit analysis.
- Let's take a simplified subset of the equations written above and express them in terms of the mass coordinate, rather than position:

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi\rho r^2}, \tag{18.2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}, \tag{18.3}$$

$$\frac{\mathrm{d}F}{\mathrm{d}m} = \varepsilon_0 \rho T^n, \tag{18.4}$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = -3\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}, \tag{18.5}$$

$$P = \frac{R_g \rho T}{\mu}. \tag{18.6}$$

- We've neglected things like radiation pressure and taken simple opacities.
- Now introduce the dimensionless mass fraction parameter

$$x = \frac{m}{M_*}, \tag{18.7}$$

so that our functions may be expressed now in terms of dimensionless functions of $x$:

$$
\begin{aligned}
r &= f_1(x) R_*, && (18.8)\\
P &= f_2(x) P_*, && (18.9)\\
\rho &= f_3(x) \rho_*, && (18.10)\\
T &= f_4(x) T_*, && (18.11)\\
F &= f_5(x) F_*. && (18.12)
\end{aligned}
$$

- Now, plugging in these scaling equations Eqs. (18.8)-(18.12) into Eqs. (18.2)-(18.6) gives

$$
\begin{aligned}
\frac{\mathrm{d}f_2}{\mathrm{d}x} &= -\frac{x}{4\pi f_1^4}; \qquad P_* = \frac{GM_*^2}{R_*^4}, && (18.13)\\
\frac{\mathrm{d}f_1}{\mathrm{d}x} &= \frac{1}{4\pi f_1^2 f_3}; \qquad \rho_* = \frac{M_*}{R_*^3}, && (18.14)\\
f_2 &= f_3 f_4; \qquad T_* = \frac{\mu P_*}{R_g \rho_*}, && (18.15)\\
\frac{\mathrm{d}f_4}{\mathrm{d}x} &= -\frac{3 f_5}{4 f_4^3 (4\pi f_1^2)^2}; \qquad F_* = \frac{ac}{\kappa}\frac{T_*^4 R_*^4}{M_*}, && (18.16)\\
\frac{\mathrm{d}f_5}{\mathrm{d}x} &= f_3 f_4^n; \qquad F_* = \varepsilon_0 \rho_* T_*^n M_*. && (18.17)
\end{aligned}
$$

- The equations above on the left are a set of nonlinear differential equations of dimensionless quantities that are **independent of the total mass**.
- The *dimensional* coefficients of the $f_i(x)$ functions, i.e., $T_*$, $P_*$, etc., are obtained as functions of mass by solving the algebraic equations on the right hand side of the above equations.
- So combining the solutions for each of these 2 procedures gives us physical profiles of quantities for any $M_*$.
- The key point is that the shapes of the profiles as a function of $x$ is the same for all stars. They only differ by a constant factor determined by the mass.
- This is called *homology* relations.

## 18.3 Dependence on mass

- The main sequence gives a tight relationship between luminosity and effective temperature, and one would like to understand this physically.
- The slope can be expressed as

$$\log L = \alpha \log T_{\mathrm{eff}} + \mathrm{const.} \qquad (18.18)$$

- Another important property is the correlation of luminosity on mass:

$$L \propto M^{\nu}. \qquad (18.19)$$

- By solving the algebraic equations given in Eqs. (18.13)-(18.17), we can ascertain the dependence of physical properties on the stellar mass.
- By plugging the density and pressure into the temperature expression, one obtains

$$T_* = \frac{\mu G}{R_g}\frac{M_*}{R_*}, \qquad (18.20)$$

and using this in the first flux equation, we find

$$F_* = \frac{ac}{\kappa}\left(\frac{\mu G}{R_g}\right)^4 M_*^3. \tag{18.21}$$

- This is an important result. At any interior point a star 10 times more massive than another will have 1000 times the flux.
- At the surface, therefore, the luminosity is

$$L \propto M_*^3, \tag{18.22}$$

which is what we set out to find.

- Using some of these results and substituting into the previous ones, we also find

$$R_* \propto M^{\frac{n-1}{n+3}}. \tag{18.23}$$

- This relates a star's radius to a star's mass depending on details of the energy generation.
- Let's consider $n \approx 4$ for proton chain stars (lower mass), and $n \approx 16$ for CNO cycle stars (higher mass).

| $n = 4$ | $n = 16$ |
|---|---|
| $R \propto M^{3/7}$ | $R \propto M^{0.8}$ |

- So in massive stars, the radius is proportional to the mass, whereas the mass dependence is weaker in less massive stars. But the relationship is not inverse like we saw for compact objects.
- Using this result and considering the density gives

$$\rho_* \propto M^{2\frac{3-n}{3+n}}. \tag{18.24}$$

- Since $n$ is always larger than 3, this results shows that lower-mass stars are denser than higher-mass stars. We've determined this already from other means.
- Now for the main sequence on the H-R diagram, the radius may be eliminated from $L = 4\pi\sigma R^2 T_{\text{eff}}^4$ using what we just found to give

$$L^{1-\frac{2(n-1)}{3(n+3)}} \propto T_{\text{eff}}^4. \tag{18.25}$$

- This gives

| $n = 4$ | $n = 16$ |
|---|---|
| $\log L = 5.6 \log T_{\text{eff}} + c$ | $\log L = 8.4 \log T_{\text{eff}} + c$ |

- This is in rough agreement with observations, except for the very low main sequence (where stars are fully convective).
- Many other interesting quantities can be computed in this way. An important one is the main sequence lifetime dependence on mass. A reasonable assumption is that the nuclear resevoir depends on mass, and the rate of consumption is related to the luminosity. So

$$\tau_{\text{MS}} \propto \frac{M}{L} \propto M^{-2}. \tag{18.26}$$

- We can estimate a minimum mass on the main sequence too.

- According to what we found already, the temperature

$$T_* \propto \frac{M_*}{R_*} \propto M^{\frac{4}{n+3}}. \tag{18.27}$$

- This holds at the center as well. For a low-mass star

$$T_c \propto M^{4/7}. \tag{18.28}$$

- The minimum temperature for hydrogen burning is about 4 million K. We know the Sun is doing this, so let's scale this equation to the Sun:

$$\frac{T_c}{T_{c,\odot}} = \left(\frac{M}{M_\odot}\right)^{4/7}. \tag{18.29}$$

- Since $T_c \geq T_{\rm min}$, we can set the condition that

$$\frac{M}{M_\odot} \geq \left(\frac{T_{\rm min}}{T_{c,\odot}}\right)^{4/7}. \tag{18.30}$$

- This gives about $M_{\rm min} \approx 0.1 M_\odot$ using the Sun's core temperature.

- The luminosity at this minimum mass, estimated similarly, is

$$\frac{L_{\rm min}}{L_\odot} = \left(\frac{M_{\rm min}}{M_\odot}\right)^3 \approx 10^{-3}. \tag{18.31}$$