Unit 15

Convection

Another important carrier of energy from the stellar interior outward is convection.

15.1 Temperature gradients ("dels")

- There are several manipulations we can carry out to make the expressions we derived more useful for later.
- For future use we will need different forms of Equation (13.8). Take hyrdrostatic equilibrium and use logarithmic derivatives:
$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r} = -\frac{Gm\rho}{rP}. \tag{15.1}$$
- Dividing both sides by $\mathrm{d}\ln T/\mathrm{d}\ln r$ gives a new quantity we'll call "del"
$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = -\frac{r^2 P}{Gm\rho}\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{15.2}$$
which is the true driving gradient in the star.
- If we now consider that the luminosity L is carried ONLY by radiation, then we can define "delrad"
$$\nabla_{\mathrm{rad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{P\kappa_{\mathrm{R}}}{T^4}\frac{L}{m}, \tag{15.3}$$
where we used Equation (13.8).
- So if $\nabla = \nabla_{\mathrm{rad}}$, then all the luminosity is radiative. If $\nabla_{\mathrm{rad}} > \nabla$, there is some other transport mechanism of the energy in addition to radiation.
- This quantity is the local slope which is required if all the luminosity were carried by radiation through diffusion.
- In fact, we will use this as a comparison in this unit to a similar quantity we've already introduced in Equation (11.14),
$$\nabla_{\mathrm{ad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{15.4}$$
where this is defined in an "adiabatic" sense, or, i.e., at constant entropy.
- The value of 0.4 comes when considering an ideal gas:
$$\nabla_{\mathrm{ad}} = \frac{\frac{5}{3} - 1}{\frac{5}{3}} = \frac{2}{5} = 0.4. \tag{15.5}$$
This is an important number to keep in mind.

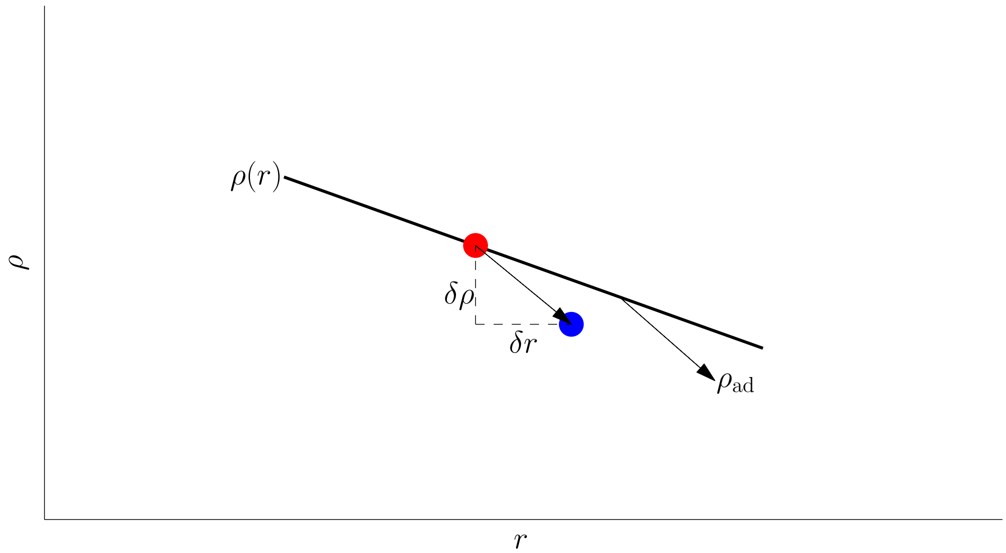

Figure 15.1: Convective instability. The curve $\rho(r)$ denotes the density gradient in some small region of a stellar interior. The arrow is the direction of an adiabat for this material. Take a parcel (red dot) in equilibrium with density ρ, and displace it upwards ($\delta r > 0$) adiabatically. It ends up where the blue dot is. This parcel now has a lower density than the surroundings ($\delta\rho < 0$), and so will continue to rise toward the surface until the conditions change (if they change). The density does not decrease sufficiently fast enough to be stable to convection.

15.2 The convective instability

- Consider in what follows an ideal gas.
- Assume a blob of gas of density ρ and pressure P at point r. It is in equilibrium with its surroundings also then of density ρ and pressure P.
- Let's displace the blob, or perturb it vertically into the medium (at $r + \delta r$) which now has density ρ' and pressure P', which we know are less than the unprimed quantities. What happens to the blob?
- Let ρ^* be the density of the blob. If $\rho^* < \rho'$ then the blob will be buoyant and continue rising: this is unstable. If $\rho^* > \rho'$ then the blob will return to its original position and there is no instablility. So how do ρ^* and ρ' compare?
- Two physically-motivated assumptions: (1) The pressure imbalances are quickly removed by acoustic waves (on the dynamical time scale), so that the pressure of the blob is also P'. (2) Heat is exchanged on the thermal timescale, which is long, so this is an adiabatic displacement.
- We know for an adiabatic displacement that $P/\rho^\gamma = \text{const}$ [Equation (11.11)]. Comparing at bottom and top we can show

$$\rho^* = \rho \left(\frac{P'}{P}\right)^{1/\gamma}. \tag{15.6}$$

- Let's expand the environmental pressure and density about point r to first order:

$$P' = P(r + \delta r) = P(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \ldots \tag{15.7}$$

$$\rho' = \rho(r + \delta r) = \rho(r) + \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r + \ldots \tag{15.8}$$

- Substitute Equations (15.7)-(15.8) into (15.6) and expand (binomial):

$$\rho^* = \rho + \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r. \tag{15.9}$$

- For an instability to occur, $\rho^* - \rho' < 0$, or

$$\rho^* - \rho' = \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r < 0. \tag{15.10}$$

- So, an instability occurs if

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} < \frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{15.11}$$

  where we introduced the adiabatic gradient

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} = \frac{1}{\Gamma}\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{15.12}$$

  where we've denoted $\gamma = \Gamma$ in the adiabatic case.

- This can be interpreted as the density gradient resulting from adiabatic motion in the given pressure gradient.

- Since the gradient of pressure is always negative (hydrostatic equilibrium), instability occurs when the density does not decrease sufficiently rapidly compared to the adiabatic case.

- See Figure 15.1 for a schematic of this.

- Note that it is convention to express Equation (15.11) as

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P} < \frac{1}{\Gamma_1}. \tag{15.13}$$

  (Note since we've divided by a negative number, $\mathrm{d}\ln P/\mathrm{d}r$, the inequality changes). For a fully ionized ideal gas, the RHS is $3/5$.

- Let's now consider the force per unit volume acting on the displaced blob. That force (buoyancy and gravitational) is $F = -(\rho^* - \rho')g$, since g acts downwards.

IN CLASS WORK

Use this force in Newton's second law and derive a simple equation of motion for the displacement δr. Show that a characterstic frequency N comes out

$$N^2 = \frac{g}{\rho *}\left(\frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = \frac{g}{\rho *}\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right], \tag{15.14}$$

called the Brunt-Väisälä frequency. Examine the solutions of the equation of motion based on the possible values of N in the stable or unstable condition.

Answer:
From Newton's second law

$$\rho^* \frac{\mathrm{d}^2 \delta r}{\mathrm{d}t^2} = -(\rho^* - \rho')g.$$

If we plug in Equation (15.10) we get the equation of motion

$$\frac{\mathrm{d}^2 \delta r}{\mathrm{d}t^2} + N^2 \delta r = 0,$$

where N is the Brunt-Väisälä frequency given above.
A general solution to this equation is $\delta r \propto \mathrm{e}^{\pm \mathrm{i} N t}$.
If the medium is stable to convection, we know that $N^2 > 0$. When this is the case, the solution is thus sinusoidal and the blob δr oscillates about a given point (gravity/buoyancy waves).
In the other case, $N^2 < 0$ and so N is imaginary: $N \rightarrow \mathrm{i}N$. The the solution goes as $\delta r \propto \mathrm{e}^{-Nt} + \mathrm{e}^{Nt}$. This solution describes an exponentially growing parcel, in other words, a convective instability.