Unit 13

Energy Transport: Radiation

Energy liberated in stellar interiors is transferred to the surface by radiation, convection, and conduction. We are not considering here radiation from a stellar photosphere, only the movement of photons in stellar interiors.

13.1 Basics

- The basic idea is that photons emitted in hot regions of a star are absorbed in cooler regions of a star, thus "transferring" energy from hot to cool.
- As we'll soon see, the "efficiency" of this transfer will depend on the temperature gradient. A very rough approximation of the gradient for the Sun is $\mathrm{d}T/\mathrm{d}r \approx (T_{\mathrm{surf}} - T_{\mathrm{core}})/(r_{\mathrm{surf}} - r_{\mathrm{core}}) \approx T_c/R_\odot \approx -2.25 \times 10^{-4}\,\mathrm{K\,cm^{-1}}$.
- Figure 13.1 shows this number compared to the "true" temperature gradient in the interior of the Sun. Clearly there is more physics taking place than we've considered so far.
- The efficiency of radiation will also depend on the ability of the photons to travel freely.
- Consider the luminosity roughly as the (total radiation energy stored in the star) divided by the (escape time for photons).
- The radiation energy is the energy density of photons (Eq. 8.4), say, at the central temperature of the star (the Sun in this case)
$$E_\gamma = aT_c^4 \cdot \frac{4\pi}{3} R_\odot^3. \tag{13.1}$$
- For the photon escape time, let's first consider that the Sun were completely transparent to photons. The time would then be $R_\odot/c = 2.32$ s. The resulting luminosity would be quite large!

> **EXAMPLE PROBLEM 13.1:** If we regard the Sun as a large cavity filled with photons, compute the luminosity by estimating the total energy stored in the radiation field and the Sun becoming completely transparent. Express the luminosity in $L_\odot$.

- More formally, the mean free path of photons can be expressed as
$$\ell_{\mathrm{ph}} = \frac{1}{\kappa\rho}, \tag{13.2}$$
where κ is some absorption coefficient (in units of cross section per unit mass) that will be given a physical meaning later, and ρ is the mass density.

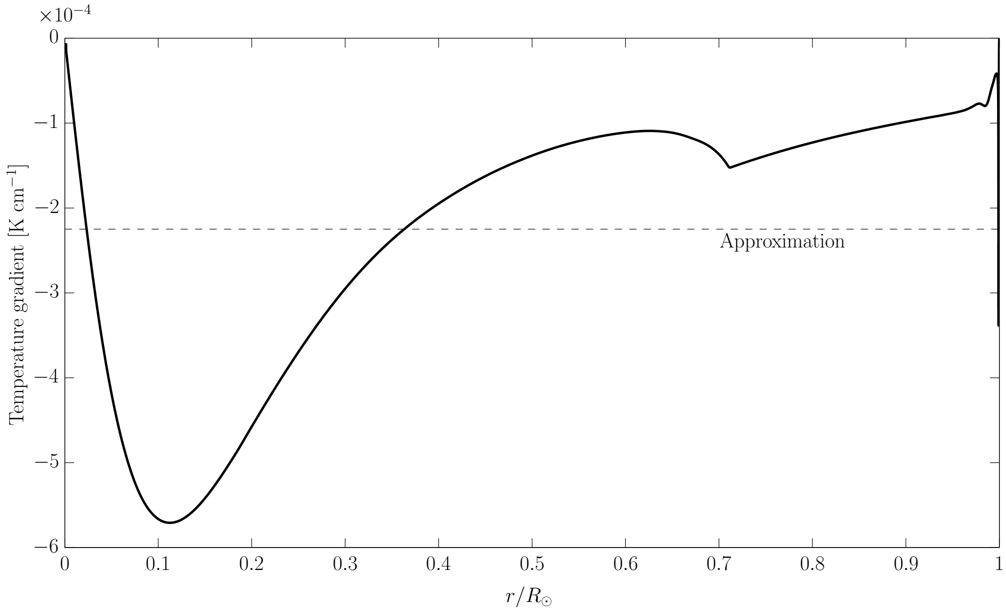

Figure 13.1: The interior temperature gradient of a solar model. Also plotted is a simple estimate of the gradient $\sim T_c/R_\odot$.

- Again for some typical interior solar values. $\kappa \approx 1\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ and simply a mean density of $\rho \approx 1.4\,\mathrm{g\,cm}^{-3}$, gives a mean free path of $\ell_{\mathrm{ph}} \approx 1$ cm, not too inconsistent with the earlier estimate, but still quite small.
- Nonetheless, radiative transport occurs by the non-vanishing net flux outward, due to the hotter material below which sets up the gradient.
- Because of the small mean free path, transport can be treated as a **diffusion process** in the interior. (Near the surface, however, this simplification starts to break down).

13.2 Diffusion

- Quick and dirty derivation of Fick's Law of diffusion, just to get the point across.
- Consider **particles** diffusing (randomly) in 3D space at some boundary r.
- Let n be the particle number density, $\overline{v}$ be the mean velocity, and ℓ the mean free path, such that $\ell = 1/\sigma n$, with σ the cross section.
- Consider isotropy. Then about 1/3 of the particles will be moving in the $\hat{r}$ direction. About 1/2 of those will be moving in the $-\hat{r}$ direction
- Flux is a quantity (like number of particles or energy) per unit area per unit time.
- From one direction, the particle flux is

$$F_+ = \frac{1}{6} n_{r-\ell}\overline{v}_{r-\ell} \tag{13.3}$$

- From the other direction

$$F_- = \frac{1}{6} n_{r+\ell}\overline{v}_{r+\ell} \tag{13.4}$$

- Net flux
$$F = F_+ - F_- = \frac{1}{6}\overline{v}(n_{r-\ell} - n_{r+\ell}), \tag{13.5}$$
assuming that $v_{r-\ell} \approx v_{r+\ell} = \overline{v}$.

- If the mean free path does not change on the scale of the density gradient, then
$$\begin{aligned} F &= \frac{1}{6}\overline{v}\left[n_{r-\ell} - n_r - (n_{r+\ell} - n_r)\right] \\ &= \frac{1}{6}\overline{v}\left[-\ell\frac{\mathrm{d}n}{\mathrm{d}r} - \ell\frac{\mathrm{d}n}{\mathrm{d}r}\right] \\ F &= -D\,\nabla_r n, \end{aligned}$$
where the diffusion coefficient $D = 1/3\,\overline{v}\,\ell$. This is Fick's Law. Again, if ℓ is large, this fails.

- This is generic. On the left you have a flux (in this case of number of particles) and on the right a gradient of density (in this case number density of particles). Note that the flux is carried from a high concentration to a low concentration of particles.

- But we want to compute the flux of diffusing radiative energy. So we need an energy density.

- For photons, we can just let $\overline{v} = c$, $\ell = \ell_{\mathrm{ph}} = 1/\kappa\rho$, and $n = u$. See Equation (8.3) and note that $u = 3P$ for a relativistic system, as derived previously, which gives
$$u = aT^4. \tag{13.6}$$

- So then the radiative flux F_{rad} is
$$F_{\mathrm{rad}} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{13.7}$$

- The local luminosity at any point passing through a sphere of radius r is $L(r) = 4\pi r^2 F_{\mathrm{rad}}$, so then rearranging we have
$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac}\frac{\kappa\rho}{r^2}\frac{L}{T^3}. \tag{13.8}$$

- This is a fundamental equation of stellar structure.

13.3 Frequency dependence of radiation

- What we just did was too simple, even in the diffusion approximation. Our answer is in fact integrated over all photon energies.

- In principle, there is a frequency dependence on the flux F_ν since the energy density and the opacity are partitioned in frequency.

- Let us go back to Equation (13.6) and instead consider
$$u_\nu = \frac{4\pi}{c}B_\nu(T), \tag{13.9}$$
where B is the Planck function for a blackbody radiator
$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{h\nu/k_{\mathrm{B}}T} - 1}. \tag{13.10}$$
This is just from our Bose-Einstein distribution function, Equation (8.1), written in terms of frequency instead of momentum.

- Also keep in mind that the integrated Planck function

$$B(T) = \int_0^\infty B_\nu(T)\,\mathrm{d}\nu = \frac{ac}{4\pi}T^4. \tag{13.11}$$

- Fick's Law now becomes

$$F_\nu = -\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\frac{\mathrm{d}B_\nu}{\mathrm{d}r} = -\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{13.12}$$

- The total flux integrated over all frequencies is then

$$F_{\mathrm{rad}} = \int F_\nu\,\mathrm{d}\nu = -\frac{4\pi}{3}\frac{1}{\rho}\frac{\mathrm{d}T}{\mathrm{d}r}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu. \tag{13.13}$$

- Comparing Equation (13.13) with Equation (13.7), we see that the κ in the latter is

$$\frac{1}{\kappa} = \frac{\pi}{ac}\frac{1}{T^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu. \tag{13.14}$$

- But since

$$\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu = \frac{\mathrm{d}}{\mathrm{d}T}\int_0^\infty B_\nu \mathrm{d}\nu = \frac{\mathrm{d}B}{\mathrm{d}T} = \frac{ac}{\pi}T^3, \tag{13.15}$$

  where $B = acT^4/4\pi$ (the integral over all frequencies), we can then define

$$\frac{1}{\kappa_{\mathrm{R}}} \equiv \frac{1}{\kappa} = \left(\int_0^\infty \frac{1}{\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu\right)\left(\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu\right)^{-1}, \tag{13.16}$$

  where κ_{R} is the *Rosseland mean opacity*.

- All this implies is that Equations (13.7) and (13.8) should replace the opacity by the Rosseland mean opacity:

$$F_{\mathrm{rad}} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\mathrm{R}}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{13.17}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac}\frac{\kappa_{\mathrm{R}}\rho}{r^2}\frac{L}{T^3}. \tag{13.18}$$

- Note that this weighted opacity gives high frequencies more weight than lower ones (as one could find by differentiating).

- Before we go onto using these expressions to understand stellar structure, let's look at a few of the major sources of κ_{R}.