Unit 10

Polytropes

10.1 Motivation and derivation

- So far we've collected 3 equations for stellar structure, collected here for convenience

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi\rho r^2, \\ \frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{G\rho m}{r^2}, \\ \frac{\mathrm{d}L}{\mathrm{d}r} &= 4\pi\rho r^2\,\varepsilon.\end{aligned}$$

- There are several others that we need to full-out model a real star. But even now we can get some very important insights on stellar structure.
- Ignore the 3rd equation for now. The first 2 equations have 3 unknowns and cannot be solved simultaneously as they stand.
- First law of thermodynamics

$$\frac{\mathrm{d}Q}{\mathrm{d}T} = \frac{\mathrm{d}U}{\mathrm{d}T} + P\frac{\mathrm{d}V}{\mathrm{d}T} = C = c_V + P\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_P. \tag{10.1}$$

- Ideal gas: $P = RT/V\mu$, where again $V = (1/\rho)$ is the specific volume and recall that $c_P - c_V = R/\mu$. We then find after manipulation

$$\frac{\mathrm{d}T}{T} + \frac{1}{n}\frac{\mathrm{d}V}{V} = 0, \tag{10.2}$$

  where n is the polytropic index $n = (c_V - C)/(c_P - c_V)$.
- We can eliminate the temperature from this to get pressure and density:

$$\frac{\mathrm{d}P}{P} = \left(1 + \frac{1}{n}\right)\frac{\mathrm{d}\rho}{\rho}, \tag{10.3}$$

  which, when integrated, gives

$$P = \text{const} \times \rho^{(1+1/n)}. \tag{10.4}$$

- A system where pressure and density are related as $P = K\rho^{1+1/n}$ is called a polytrope.

- For an adiabatic change $C = 0$, $n = 1/(\gamma - 1)$, and

$$P = K\rho^{\gamma}, \tag{10.5}$$

where $\gamma = c_P/c_V$.

- This is very useful because we can now get radial profiles of $P(r), T(r), m(r)$ and $\rho(r)$.

10.2 Lane-Emden equation

- Take hydrostatic equilibrium, divide by density, multiply by r^2, and use the mass gradient equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{10.6}$$

- Consider the polytropic equation of state

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 K\gamma\rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{10.7}$$

- Use polytropic index n and let the density be rewritten as a unitless quantity θ by

$$\frac{\rho}{\rho_c} = \theta^n, \tag{10.8}$$

where ρ_c is the central density of a model.

- Then

$$\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n. \tag{10.9}$$

- Let the coefficient (of units distance squared) be α^2, where

$$\alpha = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2} = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}. \tag{10.10}$$

(You might want to prove to yourself the above is true and the unit is distance).

- We then scale the radial coordinate

$$r = \alpha\xi. \tag{10.11}$$

- Then

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{10.12}$$

This is known as the Lane-Emden equation.

- Boundary conditions at the center must satisfy (noting Equation (10.8))

$$\theta(\xi) = 1 \quad \text{at} \quad \xi = 0, \tag{10.13}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = 0 \quad \text{at} \quad \xi = 0. \tag{10.14}$$

- Define the surface as

$$\theta(\xi) = 0 \quad \text{at} \quad \xi = \xi_1. \tag{10.15}$$

10.3 Polytrope solutions

- In what follows, a subscript n denotes the label of the polytropic index, whereas the supercript n is the quantity raised to the nth power.
- Assume a solution $\theta_n(\xi)$ can be found to Equation (10.12) for a given index n.
- Then the radius of the model is

$$R = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2} \xi_1 = \alpha\xi_1. \tag{10.16}$$

- The mass interior to $m(\xi)$ is

$$\begin{aligned} m(\xi) &= \int_0^{\alpha\xi} 4\pi r'^2 \rho \, \mathrm{d}r' = 4\pi\alpha^3\rho_c \int_0^{\xi} \xi'^2\theta^n \, \mathrm{d}\xi' & (10.17)\\ &= -4\pi\alpha^3\rho_c \int_0^{\xi} \frac{\mathrm{d}}{\mathrm{d}\xi'}\left(\xi'^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi'}\right) \mathrm{d}\xi' & (10.18)\\ &= -4\pi\alpha^3\rho_c\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} & (10.19)\\ m(\xi) &= -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n}\, \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}. & (10.20)\end{aligned}$$

- The total mass is

$$M = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}. \tag{10.21}$$

- Inspecting the mass and radius relations, the constant K is

$$K = NGM^{(n-1)/n}R^{(3-n)/n}, \tag{10.22}$$

where

$$N = \frac{(4\pi)^{1/n}}{n+1}\left[-\xi_1^{(n+1)/(n-1)}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}\right]^{(1-n)/n}. \tag{10.23}$$

- It is interesting to point out here that if K is known from some equation of state, then one can derive explicit mass-radius relationships. If that is not the case, then mass and radius must be predefined. More about this later.
- Mean density of model

$$\overline{\rho} = \frac{M}{V} = -\frac{3}{\xi_1}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1} \rho_c. \tag{10.24}$$

- Central density is then

$$\rho_c = -\frac{\xi_1}{3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^{-1}_{\xi=\xi_1} \frac{M}{4/3\pi R^3}. \tag{10.25}$$

- Central pressure

$$P_c = K\rho_c^{(n+1)/n}, \tag{10.26}$$

or

$$P_c = W_n\frac{GM^2}{R^4}, \tag{10.27}$$

where

$$W_n = \left[4\pi(n+1)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^2_{\xi=\xi_1}\right]^{-1}. \tag{10.28}$$

Note that this is the coefficient we were computing in some simple example problems (cf. Equation (9.11)).

- So pressure throughout model is

$$P = P_c\theta^{n+1}. \tag{10.29}$$

- For the temperature, assume an ideal gas with μ constant, then

$$T = T_c\theta, \tag{10.30}$$

where

$$T_c = \Theta\frac{GM\mu m_\mathrm{u}}{k_\mathrm{B}R}, \tag{10.31}$$

and

$$\Theta = \left[-(n+1)\xi_1\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}\right]^{-1}. \tag{10.32}$$

Again here, this coefficient is a number obtained through simpler means using only hydrostatic equilibrium (cf. Problem 9.1).

- The distribution of mass in a polytrope can be obtained easily (see Equation (10.20) and Equation (10.21))

$$q = \frac{m}{M} = \left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^{-1}_{\xi=\xi_1}. \tag{10.33}$$

- Only 3 analytical solutions of the Lane-Emden equation are possible:

$$n=0 \quad ; \qquad \theta_0 = 1-\frac{1}{6}\xi^2, \tag{10.34}$$

$$n=1 \quad ; \qquad \theta_1 = \frac{\sin\xi}{\xi}, \tag{10.35}$$

$$n=5 \quad ; \qquad \theta_5 = (1+\frac{1}{3}\xi^2)^{-1/2}. \tag{10.36}$$

COMPUTER PROBLEM 10.1: [40 points]: Solve the Lane-Emden Equation (10.12) numerically with the two boundary conditions using your method of choice for your assigned index n.

What to do

- To solve the 2nd-order nonlinear differential equation, which is a difficult task to do as is, the first thing you will need to do is to make a substitution to generate 2 first-order differential equations, as can always be done. A suitable choice may be to let $y = \theta$ and $z = \mathrm{d}\theta/\mathrm{d}\xi = y'$. You'll then be able to have an equation for y' and one for z'.
- Next you need to choose your solver. You can treat this as a boundary-value problem, in which case a method like Newton-Rhapson can be used. Fancy software like `IDL`, Python, and `MATLAB` have built-in boundary value algorithms, but can be tricky to implement in some cases. Perhaps a better option is to use such languages and code your own algorithm, perhaps implementing a "shooting method," which treats the problem as an initial-value problem. You "shoot" from the center, say, and work your way out to the end of the model using an integrator. A common and very handy integrator is Runge-Kutta. I'd suggest this method (shooting) because you don't need any sophisticated algorithms and it works! But it's your choice. A 4th-order Runge-Kutta is sufficient for this problem if you choose to do so. There's lots of information on this in the *Numerical Recipes* book, for example. If you do this, make sure you verify through testing that you choose the proper grid spacing.
- You have to be a little careful about how you treat the center of the model since there is a possible divergence in your equation(s) (as should be apparent already). If one expands the Lane-Emden equation about the origin, it can be shown that

$$\theta_n(\xi) \simeq 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{15120}\xi^6 + \ldots \tag{10.37}$$

  You can use this approximation to set your first values for y and z (if you choose to solve the problem with this class of methods) by taking a very small, but finite, starting ξ. Then just run it until you cross the first zero in θ_n. Make sure your grid is sufficiently fine so the solution is smooth.

What to hand in

(a) You will solve for a polytrope of given index n, assigned in the table. You will provide the instructor the values (including 3 decimal places!) you find for the table columns <u>for your n only</u>. For the last 3 columns in the table, compute those values for 1 solar mass, 1 solar radius, and composition $X = 0.7$ and $Z = 0.02$. We'll fill the table in together when finished. Also provide a copy of your code. [15 pts]

(b) Plot the following quantities on a single full-page plot with appropriate labels and clearly distinguishable lines: $\theta_n(\xi)$, $\theta_n^n(\xi)$, $\theta_n^{n+1}(\xi)$, and $q(\xi)$. What do each of the 4 quantities that you are showing here physically correspond to? [10 pts]

(c) Plot your model on a temperature-density $(T-\rho)$ plot, as in Figure 8.1 (you don't need to plot the boundary lines for different equations of state). Then overplot one of your MESA solar mass models, perhaps the one from Computer Problem 0 (the one found at the end of the MESA notes on the course webpage). If you didn't save the density or temperature interior profile, it's quick to rerun MESA agagin with those quantities included. [5 pts]

(d) Derive those first 4 terms of the expansion in Equation (10.37). <u>Hint</u>: First show or explain that if $\theta(\xi)$ is a solution of the Lane-Emden equation, then so must $\theta(-\xi)$. This motivates you to try a solution of the form below of even powers which you can plug into the Lane-Emden equation:

$$\theta(\xi) = C_0 + C_2\xi^2 + C_4\xi^4 + C_6\xi^6 + \ldots \tag{10.38}$$

Don't forget to use any boundary conditions you may need. [10 pts]

n	Name	ξ_1	$\rho_c/\overline{\rho}$	N_n	W_n	Θ_n	ρ_c [g cm^{-3}]	P_c [dyne cm^{-2}]	T_c [K]
0.0	–								
0.5	Hannah								
1.0	–								
1.5	Harrison								
2.0	Rogelio								
2.25	Annie								
2.5	Ali								
2.75	Audrey								
3.0	Mark								
3.25	Alexander								
3.5	Kelly								
3.75	Farhan								
4.0	Bryson								
4.25	Matt								
4.5	Oana								
4.75	Manny								
5.0	–								