Unit 9

Hydrostatic Equilibrium

9.1 Derivation

- Consider Figure 9.1. Take a thin mass element in a star of thickness $\mathrm{d}r$ and surface $\mathrm{d}A$ at radius r (and thus of mass $\mathrm{d}m = \rho \mathrm{d}r \mathrm{d}A$) from the center.
- The gravitational force on that mass element is

$$\mathrm{d}F_g = -\frac{G\left[\rho(r)\mathrm{d}r\mathrm{d}A\right] m(r)}{r^2}, \tag{9.1}$$

  directed radially inward.
- In equilibrium, this force is balanced by an outward pressure force acting at r and $r + \mathrm{d}r$ ($P = \mathrm{d}F/\mathrm{d}A$)

$$\mathrm{d}F_P = \left[P(r) - P(r + \mathrm{d}r)\right]\mathrm{d}A = -\frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r\mathrm{d}A, \tag{9.2}$$

  by using the definition of the derivative.
- In hydrostatic equilibirum, $\mathrm{d}F_g + \mathrm{d}F_P = 0$, and so

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2}. \tag{9.3}$$

- We will commonly see this written in vector form as

$$\boldsymbol{\nabla}_{\boldsymbol{r}} P = \rho \boldsymbol{g}. \tag{9.4}$$

- Note that the mass element in the thin shell can be expressed as

$$\mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r, \tag{9.5}$$

  or

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2. \tag{9.6}$$

- So another convenient form is

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}m} = -\rho g \frac{1}{4\pi r^2 \rho} = -\frac{Gm}{4\pi r^4} \tag{9.7}$$

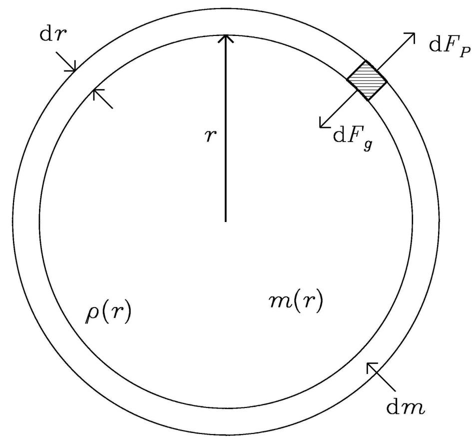

Figure 9.1: Schematic for deriving hydrostatic equilibrium. From Christensen-Dalsgaard [2003].

- Thus the mass as a function of radius is found by

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\mathrm{d}r'. \tag{9.8}$$

IN CLASS WORK

Estimate the central pressure of the Sun from the equation of hydrostatic equilibrium. Compare to the tabulated values. Try to put all final expressions in terms of scaled solar values as we have been doing.

Answer: The simplest thing one can do is ignore the derivatives in the equilibrium expression and assume the density is constant:

$$\frac{\partial P}{\partial r} = -\frac{G\rho(r)M(r)}{r^2} \tag{9.9}$$

$$\frac{P_c}{R} = \frac{3}{4\pi}\frac{GM^2}{R^5} \tag{9.10}$$

$$P_c = \frac{3}{4\pi}\frac{GM_\odot^2}{R_\odot^4}, \tag{9.11}$$

where we've replaced all values by the gross solar ones. For a general star we can show

$$P_c \approx 2.69 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \mathrm{dyne\,cm^{-2}}.$$

We know that 1 dyne cm^{-2} $= 0.1$ N m^{-2} [Pa], so we are only off in pressure by about 2 orders of magnitude when compared to the tabulated value of $2.3 \times 10^{17}\,\mathrm{dyne\,cm^{-2}}$. Not so bad actually. The tabulated value is roughly

$$P_c \approx \frac{261}{4\pi}\frac{GM_\odot^2}{R_\odot^4} \quad \text{(tabulated)}$$

PROBLEM 9.1: [10 pts]: (a) Do the same type of calculation as for the central pressure to find the central temperature and compare to the table value. Use the same mass fractions as in Problem 7.1. (b) Then, using the expressions you now have for the gas pressure and temperature, show that we can ignore the radiation pressure for the Sun in the core. I.e., show that $P_{\rm R}/P_{\rm G} \approx c(M/M_\odot)^2$, where c is a smallish number. Try to put all final expressions in terms of scaled solar values as we have been doing.

EXAMPLE PROBLEM 9.1: In Equation (9.11) we found a cheap and dirty estimate of the central pressure. Now, using Equations (9.3) and (9.6), we can find a **lower bound** for the pressure at the center of the Sun. First we need an expression for $\mathrm{d}P/\mathrm{d}m$ and then integrate from core to surface. Making a simple assumption in the integrand allows you to argue this is really a lower limit. Compare again to the previous result of the in class problem by expressing your final answer in terms of

$$P_c = \mathrm{const} \times \frac{GM_\odot^2}{R_\odot^4}, \tag{9.12}$$

where the constant is really the key quantity in your computation.

Answer:

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}.$$

We can take simply that $r = R_\odot$. Integrate both sides from the center to the surface:

$$\int_{m=0}^{m=M_\odot} \mathrm{d}P = P(M_\odot) - P(0) = -\int_{m=0}^{m=M_\odot} \frac{Gm}{4\pi r^4}\,\mathrm{d}m.$$

To obtain an absolute lower limit, let's just take $r = R_\odot$, it's the simplest thing we can do and we know it makes the overall value the smallest. Then

$$P(M_\odot) - P(0) = -\frac{G}{4\pi R_\odot^4}\int_{m=0}^{m=M_\odot} m\,\mathrm{d}m = \frac{1}{8\pi}\frac{GM_\odot^2}{R_\odot^4} < P(0).$$

This gives

$$P_c \approx 4.5\times 10^{14}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4}\,\mathrm{dyne\,cm^{-2}},$$

below the value in the table and below the one found before. But it is a lower limit. It is very interesting that this limit is valid for any star in hydrostatic equilibrium, independent of equation of state or energy production.

EXAMPLE PROBLEM 9.2: Let's improve the lower limit now (i.e., make it a bit larger). All that is needed is to assume a mean density that is a decreasing function of r such as

$$\overline{\rho(r)} = \frac{m}{4\pi r^3/3}.$$

Use this at the right step in Problem 9.1 to get a new lower limit, again expressed as

$$P_c = \mathrm{const} \times \frac{GM_\odot^2}{R_\odot^4}.$$

Answer: Picking up where we left off by solving for r as a function of the density and mass gives

$$P(0) > \frac{G}{4\pi}\int_0^{M_\odot}\frac{m}{(3m/4\pi\bar\rho)^{4/3}}\mathrm{d}m = \frac{G}{4\pi}\left(\frac{4\pi\bar\rho}{3}\right)^{4/3}\int_0^{M_\odot} m^{-1/3}\mathrm{d}m = \frac{G}{4\pi}\left(\frac{4\pi\bar\rho}{3}\right)^{4/3}\frac{3}{2}M_\odot^{2/3} = \frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^4}.$$

In the last step we simply allowed the mean density to be the gross solar values. This gives us a new value 3 times larger than the previous one, but still twice as small as that in Equation (9.11).

9.2 Simple solutions

We can solve the equation of hydrostatic equilibrium if we know the density as a function of radius or pressure. Analytically, we can make some progress from simplified assumptions.

9.2.1 Linearized density

- Let
$$\rho = \rho_c \left(1 - \frac{r}{R}\right). \tag{9.13}$$
- Then integrating Eq. 9.6 with this density profile gives
$$m(x) = \frac{4}{3}\pi R^3 \rho_c (x^3 - \frac{3}{4}x^4), \tag{9.14}$$
  where $x = r/R$.
- We know that it must be true that the total mass
$$M = m(x = 1) = \frac{\pi}{3} R^3 \rho_c, \tag{9.15}$$
  or
$$\rho_c = \frac{3M}{\pi R^3}. \tag{9.16}$$
- Using hydrostatic equilibrium and what we've just found, we can show
$$P = \frac{5}{4\pi}\frac{GM^2}{R^4}\left(1 - \frac{24}{5}x^2 + \frac{28/5^{3}}{x} - \frac{9}{5}x^4\right). \tag{9.17}$$
- Note how the coefficient is larger than before at the center ($x = 0$)

9.2.2 Isothermal atmosphere

- Take hydrostatic equilibrium in one dimension:
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g. \tag{9.18}$$
- For an incompressible fluid (constant density)
$$P = P_0 - \rho g r, \tag{9.19}$$
  where P_0 is the pressure at $r = 0$. An incompressible fluid at rest increases linearly with depth.
- Also, using the equation of state and an **isothermal** ideas gas, we can replace P in the hydrostatic equation with ρ:
$$\frac{RT}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho g, \tag{9.20}$$
  whose solution is
$$\rho = \rho_0 \exp\left(-\frac{r}{H}\right), \tag{9.21}$$
  where the (density) scale height $H = RT/\mu g$.