Unit 8

Density-temperature equation of state landscape

8.1 Radiation pressure

First consider one more equation of state.

- Particles are not the only source of pressure in a star. The radiation field of photons can also exert a pressure.
- Photons have momentum and can exchange that momentum with other objects, creating pressure.
- We already have an expression for the general pressure in Equation (6.5).
- With a degeneracy factor $g = 2$ (photons have 2 spin states, or polarizations, each with the same energy at fixed frequency), chemical potential (bosons) $\mu_c = 0$, $E = pc$, and $E_j = 0$, the distribution function Equation (6.1) is
$$n(p) = \frac{2}{h^3}\frac{1}{\exp\left(pc/kT\right) - 1}. \tag{8.1}$$
- From Equation (6.5), we thus have
$$P_{\rm rad} = \frac{8\pi c}{3h^3}\int_0^\infty \frac{p^3}{e^{pc/kT} - 1}\,{\rm d}p, \tag{8.2}$$
where we made the substitution $v \equiv c$ for photons.
- The integral can be solved (Problem 8.1) to give
$$P_{\rm rad} = \frac{1}{3}a\,T^4, \tag{8.3}$$
where $a = 4\sigma/c = 7.5 \times 10^{-15}\,{\rm erg\,cm^{-3}\,K^{-4}}$.
- Similarly as before, the energy density
$$u_{\rm rad} = aT^4 = 3P_{\rm rad}. \tag{8.4}$$
Note the similar form to relativistic, degenerate matter.

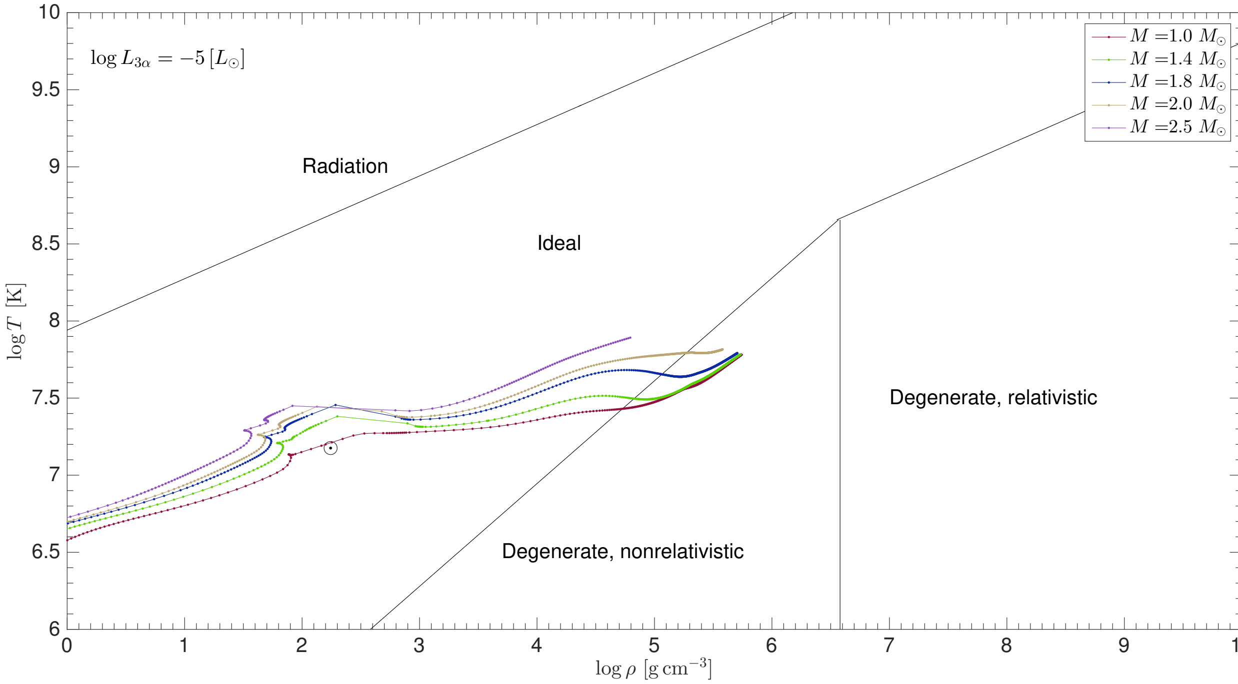

Figure 8.1: Stellar matter conditions. These boundary regimes are computed using $\mu_\mathrm{e} = 2$ and $\mu = 0.5$. A few example model tracks (central values as a funciton of time) are shown for different masses, until the point when the contribution to the luminosity by the triple$-\alpha$ process is 1 part in 10^5 solar luminosities, as denoted in the figure. In practice, these stars are near their respective tip of the red-giant branch. The current location of the Sun is given by its symbol. The radiation pressure boundary is computed from $P_\mathrm{rad} = 10 P_\mathrm{ideal}$.

PROBLEM 8.1: [10 pts]: Carry out the integral in Equation (8.2) to show that

$$a = \frac{8\pi^5}{15}\frac{k^4}{h^3c^3}. \tag{8.5}$$

Hint: make the substitution $x = pc/kT$.

8.2 Putting it all together

- Putting the previous sections together, the pressure of stellar matter through the equation of state in general is
$$P = P_\mathrm{ion} + P_\mathrm{e} + P_\mathrm{rad}. \tag{8.6}$$
- In some cases, the electron pressure is from degenerate particles. In rare cases, the ions can become degenerate too.
- Not all of these pressure terms contribute equally to the total pressure at any given time, as you saw in Problem 7.1.
- Consider the total gas pressure of an ideal gas as
$$P_\mathrm{gas}^\mathrm{ideal} = P_\mathrm{ion} + P_\mathrm{e} = \frac{\rho k_\mathrm{B} T}{\mu m_\mathrm{u}}. \tag{8.7}$$
- It is useful to compare regions where this and $P_\mathrm{e}^\mathrm{deg}$ and P_rad compete and transition to one another.

- First consider where an ideal gas transitions to a degenerate nonrelativistic one. Equating Equation (7.10) and Equation (8.7) gives

$$\frac{\rho}{\mu_{\rm e}^{5/2}} = \left(\frac{k_{\rm B}}{C\mu m_{\rm u}}\right)^{3/2} T^{3/2}, \tag{8.8}$$

  where C is the constant prefactor in Equation (7.10) .

- For large densities or low temperatures, i.e., when $\rho T^{-3/2} >$ const, the gas is dominated by degenerate pressure. This is shown by the line of a slope of 2/3 in Figure 8.1.

- When electron speeds become relativistic, Equation (7.13) becomes appropriate, and when equated with the ideal gas pressure yields

$$\frac{\rho}{\mu_{\rm e}^{4}} = \left(\frac{k_{\rm B}}{D\mu m_{\rm u}}\right)^{3} T^{3}, \tag{8.9}$$

  where D is the constant prefactor in Equation (7.13).

- This is shown in Figure 8.1 with the line of slope 1/3 at high temperature and density.

- In the degenerate regime, the transition from non- to relativistic is found by equating Equation (7.10) and Equation (7.13), which is independent of temperature (since these are completely degenerate systems):

$$\frac{\rho}{\mu_{\rm e}} = \left(\frac{D}{C}\right)^{3}. \tag{8.10}$$

  This is given in the figure by the vertical line.

- Finally, we can determine where radiation pressure starts to exceed ideal gas pressure. We use Equation (8.3) to find

$$\frac{\rho}{\mu} = \frac{1}{3}\frac{a m_{\rm u}}{k_{\rm B}} T^{3}. \tag{8.11}$$

- In Figure 8.1 this is shown by the line at the bottom right of slope 1/3.